Degree Spectra of Homeomorphism Types of Polish Spaces

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Abstract

A Polish space is not always homeomorphic to a computably presented Polish space. In this article, we examine degrees of non-computability of presenting homeomorphic copies of Polish spaces. We show that there exists a $\mathbf{0}'$-computable low$_3$ Polish space which is not homeomorphic to a computable one, and that, for any natural number n, there exists a Polish space X_n such that exactly the high$_{2n+3}$-degrees are required to present the homeomorphism type of X_n. We also show that no compact Polish space has an easiest presentation with respect to Turing reducibility.

Key words: computable topology, computable presentation, computable Polish space, degree spectrum.

1 Introduction

How difficult is it to describe an explicit presentation of an abstract mathematical structure? Only the isomorphism type of a structure is given to us, and then our task is to present its representative whose underlying set is (an initial segment of) the natural numbers ω. However, an isomorphism type of a structure does not necessarily have a computable presentation. In such a case, our next task is to determine how incomputable it is to present a representative of the isomorphism type. This has long been one of the fundamental questions in computable structure theory, and researchers in this area have obtained a huge number of interesting results on Turing degrees of presentations of isomorphism types of groups, rings, fields, linear orders, lattices, Boolean algebras, and so on, cf. [1, 8, 10, 14].

In this article we focus on presentations of Polish spaces. The notion of a presentation plays a central role, not only in computable structure theory, but also in computable

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analysis [3, 4, 27]. In this area, one of the most crucial problems was how to present large mathematical objects (which possibly have the cardinality of the continuum) such as metric spaces, topological spaces and so on, and then researchers have obtained a number of reasonable answers to this question. In particular, the notion of a computable presentation of a Polish space has been introduced around 1950-60s, cf. [20], and since then this notion has been widely studied in several areas including computable analysis [3, 23, 27] and descriptive set theory [21].

In recent years, several researchers succeeded to obtain various results on Turing degrees of presentations of *isometric isomorphism types* of Polish spaces, separable Banach spaces, and so on, cf. [7, 18, 19]. However, most of works are devoted to metric structures, and there seem almost no works on presentations on homeomorphism types of Polish spaces. The investigation of Turing degrees of *homeomorphism types* of topological spaces (not necessarily Polish) was initiated in [25] in analogy with the earlier investigation of degrees of isomorphism types of algebraic structures. Some results were obtained for domains but the case of Polish spaces was apparently not investigated seriously so far.

Every Polish space is homeomorphic to the Cauchy completion of a metric on (an initial segment of) the natural numbers ω, so one may consider any distance function $d\colon \omega^2 \to \mathbb{Q}$ as a presentation of a Polish space. Then, observe that there are continuum many homeomorphism types of Polish spaces. In particular, by cardinality argument, there is a Polish space which is not homeomorphic to any computably presented Polish space. Surprisingly however, it was unanswered until very recently even whether the following holds:

Question 1. *Does there exist a $0'$-computably presented Polish space which is not homeomorphic to a computably presented one?*

Note that countable spaces are useless to solve this problem because of the "hyperarithmetic-is-recursive" phenomenon, cf. [11]; see also Section 2.3. The solution to Question 1 was very recently obtained by the authors of this article, and independently by Harrison-Trainor, Melnikov, and Ng [13]. One possible approach to solve this problem is using Stone duality between countable Boolean algebras and zero-dimensional compact metrizable spaces (where note that compact metrizable spaces are always Polish); see also Section 2.4. Combining this idea with classical results on isomorphism types of Boolean algebras [17], one can conclude that every *low_4-presented* zero-dimensional compact metrizable space is homeomorphic to a computable one. This is also noticed by the authors of this article and independently by Harrison-Trainor, Melnikov, and Ng [13].

Our next step is to develop new techniques other than Stone duality. More explicitly, the next question is whether there exists a Polish space whose homeomorphism degree spectum is different from that of a zero-dimensional compact Polish space. Here, by the homeomorphism degree spectrum of a space X we mean the collection of Turing degrees which compute a presentation of a homeomorphic copy of X. In particular, it is natural to ask the following:

Question 2. *Does there exist a low_4-presented Polish space which is not homeomorphic to a computably presented one?*

One of our main results in this article is that there exists a $0'$-computable *low_3* infinite dimensional compact metrizable space which is not homeomorphic to a computable one.

This solves Question 2. By using similar techniques, we also construct, for any $n \in \omega$, an infinite dimensional compact metrizable space X_n whose homeomorphism degree spectrum is the high$_{2n+3}$-degrees; that is, X_n is homeomorphic to a $\mathbf{d}$-computably presented Polish space if and only if $\mathbf{d}$ is high$_{2n+3}$. This also clarifies substantial differences between zero-dimensional compact metrizable spaces and infinite dimensional ones since the class of high$_n$-degrees is never the degree spectrum of a Boolean algebra [15].

We prove the following results:

- For every degree $\mathbf{d}$ and every $n > 0$, there exists a space $\mathcal{Z}_{\mathbf{d},n}$ whose compact degree spectrum is $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}^{(2n-1)}\}$ (Theorems 3.3, 3.9, 3.11, 3.13).
- For every degree $\mathbf{d}$ and every $n > 0$, there exists a space $\mathcal{P}_{\mathbf{d},n}$ whose compact degree spectrum is $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}^{(2n)}\}$ and Polish degree spectrum is $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}^{(2n+1)}\}$ (Theorems 3.16, 3.18, 3.20).

Another important question is whether a given Polish space has the least Turing degree in its homeomorphism degree spectrum. In other words, it is natural to ask if the homeomorphism type of a Polish space has an easiest presentation. For instance, it is known that the isomorphism types of linear orders, trees, abelian p-groups, etc. have no easiest presentation whenever they are not computably presentable, cf. [10].

Question 3. *Does there exist a homeomorphism type of a Polish space which is not computably presentable, but have an easiest presentation with respect to Turing reducibility?*

We partially answers Question 3 in negative. More precisely, we show the cone-avoidance theorem for *compact* Polish spaces, which states that, for any non-c.e. set $A \subseteq \omega$, every compact Polish space has a presentation that does not enumerate A.

2 Preliminaries

Basic terminologies and results on computability theory and computable structure theory are summarized in [1]. For basics on computable analysis, we refer the reader to [2, 4, 3, 27]. For some basic definitions and facts on general topology and dimension theory, see also [26].

2.1 Presentations of Polish spaces

A *Polish presentation* (or simply a *presentation*) of a Polish space X is a distance function d on ω whose Cauchy completion is homeomorphic to X. Some researchers use a slightly different definition, but these definitions are equivalent: Let $a_n \in X$ be the image of $n \in \omega$ under such a homeomorphism. Then, $(a_n)_n$ is a dense sequence in X, and $d_X(a_i, a_j) = d(i, j)$ is the restriction of the metric on X to the dense set $\{a_n\}_n$. We often use the same symbol d to denote the metric d_X on X. For discussion on presentations of Polish spaces, see also [12].

A finite union of rational open balls (i.e., balls of the form $B(a_i; r)$ for some rational r) is called a rational open set. A code of a finite rational open cover of X is a finite set

$E \subseteq \omega \times \mathbb{Q}$ such that for any $x \in X$ there is $(i, r) \in E$ such that $d(x, a_i) < r$. If X is compact, then a *compact presentation* of X is a presentation of X equipped with an enumeration of a collection C of codes of all finite rational open covers of X. In particular, a compact presentation contains an information of total boundedness; that is, a function ℓ such that, given s, the 2^{-s}-net $\{B(a_n; 2^{-s}) : n < \ell(s)\}$ formally covers the whole space X.

For a Turing degree $\mathbf{d}$, a $\mathbf{d}$-*computable Polish space* is a Polish space which has a $\mathbf{d}$-computable presentation, and such a space is $\mathbf{d}$-*computably compact* if it has a $\mathbf{d}$-computable compact presentation. For a Polish space X, the *Polish degree spectrum* is the set of all Turing degrees $\mathbf{d}$ such that X has a $\mathbf{d}$-computable Polish presentation. If such a space X is compact, the *compact degree spectrum* is the set of all Turing degrees $\mathbf{d}$ such that X has a $\mathbf{d}$-computable compact presentation.

Lemma 2.1. *Let X be a compact metrizable space. If X has a $\mathbf{d}$-computable Polish presentation, then X has a $\mathbf{d}'$-computable compact presentation.*

Proof. By compactness of X, one can observe that E is a code of a finite rational open cover of X if and only if there exists $\varepsilon > 0$ such that for all $x \in X$ we have $d(x, a_i) < r - \varepsilon$ for some $(i, r) \in E$. The latter is equivalent to the existence of $s \in \omega$ such that for all $k \in \omega$ we have $d(a_k, a_i) < r - 2^{-s}$ for some $(i, r) \in E$. As E is finite, this is a Σ^0_2 condition relative to a Polish presentation of X. Hence, if X has a $\mathbf{d}$-computable Polish presentation, then the set of codes of all finite rational open covers is $\mathbf{d}'$-c.e. In other words, X has a $\mathbf{d}'$-computable compact presentation. $\square$

We will see in Section 5 a more precise relation between Polish and compact presentations.

There are another equivalent definitions of Polish and compact presentations. We may assume that X is a subspace of Hilbert cube $Q := [0, 1]^\omega$ w.r.t. the standard metric d on Q. The set $Q^\circ := (0, 1)^\omega \subset Q$ is called the pseudo-interior of Q. One can always assume that a compact metrizable space X is embedded into the pseudo-interior of Q. In particular, for each open ball B in X, there is an open ball B_* in Q with the same center and radius with B. Then, B is approximated by $B_s := B_* \cap X_s$. One can decide if $B_s \cap B_t = \emptyset$, $B_s \subseteq B_t$, etc. By $\overline{A}$ we mean the topological closure $\mathrm{cl}_Q(A)$ of A in Q.

Hyperspaces. We will also use another characterization of Polish and compact presentations of compact Polish spaces, by considering hyperspaces of compact subsets of Q.

Let $\mathcal{V}(Q)$ be the space of compact sets endowed with the lower Vietoris topology. A subbasis is given by $\{K \subseteq Q : K \cap B \neq \emptyset\}$, where B is a rational ball in Q.

Let $\mathcal{K}(Q)$ be the space of compact sets endowed with the Vietoris topology. A subbasis is given by a subbasis for the lower Vietoris topology, together with $\{K \subseteq Q : K \subseteq U \neq \emptyset\}$, where U is a rational open set in Q. This space can be equivalently obtained by endowing it with the Hausdorff metric.

In any such space, a compact set K is $\mathbf{d}$-computable if $\mathbf{d}$ computes an enumeration of the basic neighborhoods of K. We say that K is $\mathbf{d}$-computably compact if K is a $\mathbf{d}$-computable element of $\mathcal{K}(Q)$, and that K is $\mathbf{d}$-computably overt if K is a $\mathbf{d}$-computable element of $\mathcal{V}(Q)$. In particular, $K \subseteq Q$ is $\mathbf{d}$-computably compact if and only if K is $\mathbf{d}$-computably overt and $K \in \Pi^0_1(\mathbf{d})$.

We now come to the announced characterization.

Proposition 2.2. *A compact Polish space X has a $\mathbf{d}$-computable Polish presentation if and only if it has a $\mathbf{d}$-computable copy in $\mathcal{V}(Q)$.*

A compact Polish space X has a $\mathbf{d}$-computable compact presentation if and only if it has a $\mathbf{d}$-computable copy in $\mathcal{K}(Q)$.

Proof. It is the effective version of Theorem 4.14 in [16]. The only modification is the following: if U is an effective open subset of Q and F its complement, then the function $d(x, F)$ is not computable in general. However, there exists a computable function $f(x)$ which is null exactly on F. Thus the function f should be used in place of $d(x, F)$. □

2.2 Covering dimension

Let $\mathcal{U}$ be a cover of a topological space X. A *refinement* $\mathcal{V}$ of a cover $\mathcal{U}$ of X is a cover such that every $V \in \mathcal{V}$ is included in some $U \in \mathcal{U}$. If we moreover require $\overline{V} \subseteq U$, then we call it a strict refinement. Given a compact presentation of X, one can obtain a refinement sequence $(\mathcal{U}_s)_s$ of finite open coverings of X such that each $U \in \mathcal{U}_s$ is a rational open ball, and $U \in \mathcal{U}_s$ implies $X \cap U \neq \emptyset$. Let X_s be the closure of $\bigcup \mathcal{U}_s$. This gives a decreasing sequence (X_s) of rational open sets in Q such that $X = \bigcap_s X_s$. One can assume that $(\mathcal{U}_s)_s$ is a strict refinement sequence; that is, for any $U \in \mathcal{U}_{s+1}$ there is $V \in \mathcal{U}_s$ such that $\overline{U} \subseteq V$ and that the closure of the union of $\mathcal{U}_{s+1}$ is included in the union of $\mathcal{U}_s$ (by enlarging every $U \in \mathcal{U}_{s+1}$ by $1+2^{-2s}$ times). One can moreover assume that for any $U, V \in \mathcal{U}_s$, $U \cap V \neq \emptyset$ if and only if $\overline{U} \cap \overline{V} \neq \emptyset$. This assumption also ensures that $\overline{X}_{s+1} \subseteq X_s$. In summary:

Observation 2.3. *Given a compact presentation of $X \subseteq Q^\circ$, one can effectively obtain a sequence $(X_s)_{s\in\omega}$ of rational open subsets of Q such that $\overline{X}_{s+1} \subseteq X_s$ and $X = \bigcap_s X_s$.*

The order of $\mathcal{U}$ is the least number n such that any $x \in X$ is contained in at most n many sets in $\mathcal{U}$, if such a number exists. The *covering dimension* of X (written $\dim(X)$) is the least number n such that every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ of order at most $n+1$. If X is normal, it is known that such a $\mathcal{V}$ can be a strict refinement. Moreover, if X is a compact metric space, one can moreover assume that $\mathcal{V}$ consists of rational open sets. To see this, let $\mathcal{V} = (V_i)_{i\in I}$ be given, and δ be a Lebesgue number of $\mathcal{V}$. Then, consider any finite open cover $\mathcal{D} = (D_j)_{j\in J}$ of X consisting of rational open balls whose diameters are at most δ, and define $V_i^* = \bigcup\{D_j : D_j \subseteq V_i\}$. By the property of δ, every D_j is contained in some V_i, so $\mathcal{V}^* = (V_i^*)_{i\in I}$ is a cover of X. Moreover, the order of $\mathcal{V}^*$ is less than or equal to $\mathcal{V}$; that is, $\mathcal{V}^*$ is the desired one. Similarly, one can also assume that $\mathcal{U}$ is a rational open cover. In summary, $\dim(X) < n$ if and only if every rational open cover of X has a strict refinement of order at most $n+1$ consisting of rational open sets. We say that a finite collection $\mathcal{V}$ is a *finite modification* of $\mathcal{U}_s$ if every member of $\mathcal{V}$ is a finite union of elements of $\mathcal{U}_s$, and moreover $\bigcup \mathcal{V} = \bigcup \mathcal{U}_s$. The above argument shows that, if a compact presentation of X is fixed, then $\dim(X) < n$ if and only if for any s there is t such that a finite modification of $\mathcal{U}_t$ refines $\mathcal{U}_s$ and the order of $\mathcal{U}_t$ is at most n.

Lemma 2.4. *Given a compact presentation of a zero-dimensional compact metrizable space X, one can effectively find a computable pruned tree $T \subseteq 2^{<\omega}$ such that X is homeomorphic to $[T]$.*

Proof. By using a compact presentation of X, we construct a sequence of finite clopen coverings $\mathcal{C}_n$ of X such that $\mathcal{C}_n$ is pairwise disjoint and the mesh of $\mathcal{C}_n$ is at most 2^{-n}. Given $\mathcal{C}_n$, one can effectively find a finite 2^{-n-1}-net $\mathcal{U}_{n+1}$ of X. As X is zero-dimensional, by the above argument, one can find a refinement $\mathcal{C}_{n+1}$ of $\mathcal{U}_{n+1}$ such that the order of $\mathcal{C}_{n+1}$ is at most 1; that is, the collecton $\mathcal{C}_{n+1}$ is pairwise disjoint. Since $\mathcal{C}_{n+1}$ is finite, this means that $\mathcal{C}_{n+1}$ is a clopen cover of X. Clearly the sequence $(\mathcal{C}_n)_n$ yields a finite branching pruned tree T. That is, each member $\mathcal{C}_n$ is assigned to a node in T of length n, and then X is homeomorphic to the compact space $[T]$ consisting of infinite paths through T. Moreover, T is computably bounded relative to a presentation of X; that is, one can effectively compute the number of immediate successors of T. Hence, one can effectively find $S \subseteq 2^{<\omega}$ such that $[S]$ is homeomorphic to $[T]$. □

2.3 Cantor-Bendixson derivative

Let P be a topological space. The *Cantor-Bendixson derivative* of P is the set P' of all non-isolated points in P.

Lemma 2.5. *Let X be a compact Polish space.*

If $X \subseteq Q$ is $\mathbf{d}$-computably overt then X' is $\Pi^0_1(\mathbf{d}')$ and $\mathbf{d}''$-computably overt.

Therefore, if X has a $\mathbf{d}$-computable Polish presentation, then its Cantor-Bendixon derivative X' has a $\mathbf{d}''$-computable compact presentation.

Proof. Assume that X has a $\mathbf{d}$-computable Polish presentation, or equivalently that $X \subseteq Q$ is $\mathbf{d}$-computably overt.

Let $B = B(a_i; r)$ be an open ball in X. We claim that $B \cap X'$ is nonempty if and only if there is s such that $B^s = B(a_i; r - 2^{-s})$ contains infinitely many points. For the forward direction, choose $x \in B \cap X'$. Then, $d(a_i, x) < r$, so for any sufficiently large s, $d(a_i, x) < r - 2^{-s}$, i.e., $x \in B^s$. Since B^s is open and x is of rank 1, it is clear that B^s contains infinitely many points. For the backward direction, if B^s contains infinitely many points, but $B \cap X'$ is empty, then B consists of infinitely many isolated points, and so is the closure $\overline{B^s}$ of B^s as $B^s \subseteq \overline{B^s} \subseteq B$; however this is impossible by compactness. By this claim, the property $B \cap X' \neq \emptyset$ is $\Sigma^0_3(\mathbf{d})$, or equivalently $\Sigma^0_1(\mathbf{d}'')$. This means that X' is $\mathbf{d}''$-computably overt.

Next, let A be the set of all (i, s) such that $d(a_i, a_j) \geq 2^{-s}$ for any $j \neq i$. Then, A is $\mathbf{d}'$-computable. One can easily see that x is isolated in X if and only if $x \in B(a_i; 2^{-s})$ for some $(i, s) \in A$. Thus, the set of isolated points is a $\Sigma^0_1(\mathbf{d}')$ subset of X; hence X' is $\Pi^0_1(\mathbf{d}')$ in X.

By Proposition 2.2, we conclude that X' has a $\mathbf{d}''$-computable compact presentation. □

We next note that countable topological spaces are completely useless for constructing nontrivial degree spectra inside the hyperarithmetical hierarchy. Let ω_1^x be the least ordinal which is not computable in x.

Observation 2.6. *For any countable ordinal α, there is a compact metrizable space $\mathbf{O}_\alpha$ whose compact and Polish degree spectrum are both $\{x : \alpha < \omega_1^x\}$.*

Proof. Let $\mathbf{O}_\alpha$ be the compact metrizable space $\omega^\alpha+1$ endowed with the order topology. It is obvious that, if α is $\mathbf{x}$-computable, then $\mathbf{O}_\alpha$ has an $\mathbf{x}$-computable compact presentation. Conversely, if $\omega^\alpha+1$ has an $\mathbf{x}$-computable Polish presentation, then by Lemma 2.1, it has an $\mathbf{x}'$-computable compact presentation. In particular, there is a countable $\Pi^0_1(\mathbf{x}')$ class P which is homeomorphic to $\omega^\alpha+1$. Since the Cantor-Bendixson rank of $\omega^\alpha+1$ is α, and the Cantor-Bendixson rank is a topological invariant, the rank of P is also α. However, as noted by Kreisel, the Spector boundedness principle implies that the rank of a countable $\Pi^0_1(\mathbf{x}')$ class must be $\mathbf{x}'$-computable; see also [6, Section 4]. As an $\mathbf{x}$-hyperarithmetic ordinal is always $\mathbf{x}$-computable, this implies that $\alpha < \omega_1^{\mathbf{x}}$. □

This completely characterizes the compact and Polish degree spectra of countable compact metrizable spaces since every countable compact metrizable space is homeomorphic to the ordinal space $\omega^\alpha \cdot n+1$ for some $\alpha < \omega_1$ and $n \in \omega$ by Mazurkiewicz-Sierpiński's theorem. For more details, see also [11].

2.4 Stone duality

Here we show that spectra of compact zero-dimensional spaces are closely related to spectra of Boolean algebras. This follows from an effectivization of Stone duality in [22].

Let $\mathbf{B}$ be the category formed by the Boolean algebras as objects and the $\{\vee, \wedge, ^-, 0, 1\}$-homomorphisms as morphisms. Recall that a *Stone space* is a compact topological space X such that for any distinct $x, y \in X$ there is a clopen set U with $x \in U \not\ni y$ (i.e., zero-dimensional and T_1). Let $\mathbf{S}$ be the category formed by the Stone spaces as objects and the continuous mappings as morphisms.

The *Stone duality* states the dual equivalence between the categories $\mathbf{B}$ and $\mathbf{S}$. More explicitly, the Stone space $s(B)$ corresponding to a given Boolean algebra B is formed by the set of prime filters of B with the base of open (in fact, clopen) sets consisting of the sets $\{F \in s(B) \mid a \in F\}$, $a \in B$. (Note that one could equivalently take ideals in place of filters.) Conversely, the Boolean algebra $b(X)$ corresponding to a given Stone space X is formed by the set of clopen sets (with the usual set-theoretic operations). By Stone duality, any Boolean algebra B is canonically isomorphic to the Boolean algebra $b(s(B))$ (the isomorphism $f : B \to b(s(B))$ is defined by $f(a) = \{F \in s(B) \mid a \in F\}$), and any Stone space X is canonically homeomorphic to the space $s(b(X))$.

Restricting the Stone duality to the countable Boolean algebras, we obtain their duality with the compact zero-dimensional countably based spaces, and in fact with the compact subspaces of the Cantor space 2^ω. As the nonempty closed subsets of 2^ω coincide with the sets $[T]$ of infinite paths through a pruned tree $T \subseteq 2^\omega$, we obtain a close relation between such subspaces and countable Boolean algebras.

Fact 2.7. (1) *A Boolean algebra has a $\mathbf{d}$-c.e. (resp. $\mathbf{d}$-co-c.e., $\mathbf{d}$-computable) copy if and only if it is isomorphic to the Boolean algebra of clopen subsets of $[T]$ for some $\mathbf{d}$-co-c.e. (resp. $\mathbf{d}$-c.e., $\mathbf{d}$-computable) pruned tree T.*

(2) *Every* $\mathbf{d}$*-co-c.e. Boolean algebra is isomorphic to a* $\mathbf{d}$*-computable Boolean algebra.*

(3) *There is a* $\mathbf{d}$*-c.e. Boolean algebra which is not isomorphic to a* $\mathbf{d}$*-computable Boolean algebra.*

Proof. The first item follows from [22, Lemma 3]; see also [24]. The second item follows from [22, Theorem]. The third item follows from [9]. □

As already noticed by Harrison-Trainer, Melnikov, and Ng [13], one can use Stone duality to show several results on degree spectra of zero-dimensional compacta. For instance, Stone duality can be used to show the following:

Fact 2.8 (see [13])**.** (1) *There exists a zero-dimensional compact metrizable space which has a* $0'$*-computable Polish presentation, but not homeomorphic to a computable Polish space.*

(2) *If a zero-dimensional compact metrizable space has a* low$_4$ *Polish presentation, then it is homeomorphic to a computable Polish space.*

Here we prove another consequence on Stone duality. Let $\mathbf{BA}$ and $\mathbf{CP}_0$ be the classes of countable Boolean algebras and of the compact zero-dimensional Polish spaces, respectively. Let $Sp(\mathbf{BA})$, $Sp(\mathbf{CP}_0)$, $Sp_c(\mathbf{CP}_0)$ denote respectively the classes of spectra of Boolean algebras, $\mathbf{CP}_0$-spaces w.r.t. Polish presentation, and of $\mathbf{CP}_0$-spaces w.r.t. compact presentation.

Theorem 2.9. $Sp(\mathbf{BA}) = Sp_c(\mathbf{CP}_0)$.

Proof. We show that for any countable Boolean algebra B, $Sp(B) = Sp(s(B))$, and for any $X \in \mathbf{CP}_0$, $Sp(X) = Sp(b(X))$. It suffices to check the first equality because the second one follows by Stone duality. If B is a $\mathbf{d}$-computable Boolean algebra, let $T \subseteq 2^{<\omega}$ be a $\mathbf{d}$-computable pruned tree such that $b([T])$ is isomorphic to B, hence $s(B)$ is homeomorphic to $[T]$. If $\{\tau_0, \tau_1, \dots\}$ is a $\mathbf{d}$-computable enumeration of T, let $x_i \sqsupseteq \tau_i$ be the leftmost branch of T. Then $\{x_0, x_1, \dots\}$ is a $\mathbf{d}$-computable dense sequence in $[T]$, hence $Sp(B) \subseteq Sp(s(B))$. It is clear that $[T]$ is $\mathbf{d}$-computably compact.

For the converse inclusion, assume that $s(B)$ has a $\mathbf{d}$-computable compact presentation. By Lemma 2.4, $s(B)$ is homeomorphic to the subspace $[T]$ of the Cantor space for some $\mathbf{d}$-computable pruned tree $T \subseteq 2^{<\omega}$. By Fact 2.7 (1), $b([T])$ (hence also B) is $\mathbf{d}$-computably presentable. □

The Stone dual of the Cantor-Bendixon derivative is known as the Frechét derivative B' of a Boolean algebra B which is the quotient of B by the ideal generated by atoms (minimal non-zero elements). Since the isolated points x of the space $s(B)$ (realized as $[F]$ above) are precisely the atoms $[\tau] \cap [F]$ for suitable prefix $\tau \sqsubseteq x$, we obtain the following.

Proposition 2.10. *For any countable Boolean algebra* B*,* $s(B')$ *is homeomorphic to* $s(B)'$*.*

Precise complexity estimations for the Frechet derivative were obtained in [22]: a countable Boolean algebra C is $\mathbf{d}''$-computably presentable iff C is isomorphic to B' for some

$\mathbf{d}$-computable Boolean algebra B, and there is a $\mathbf{d}$-computable Boolean algebra B such that B' is not $\mathbf{d}'$-computably presentable. The iterated version is also known for any $n > 0$: a countable Boolean algebra C is $\mathbf{d}^{(2n)}$-computably presentable iff C is isomorphic to the nth derivative $B^{(n)}$ for some $\mathbf{d}$-computable Boolean algebra B, and there is a $\mathbf{d}$-computable Boolean algebra B such that the nth derivative $B^{(n)}$ is not $\mathbf{d}^{(2n-1)}$-computably presentable.

Theorem 2.9 and Proposition 2.10 imply that Lemma 2.5 is almost optimal:

Theorem 2.11. *For any $n > 0$, a space $Y \in \mathbf{CP}_0$ has a $\mathbf{d}^{(2n)}$-computable compact presentation if and only if Y is homeomorphic to the nth derivative $X^{(n)}$ for some $\mathbf{d}$-computable compact $X \in \mathbf{CP}_0$, and there is a $\mathbf{d}$-computable compact $X \in \mathbf{CP}_0$ such that the nth derivative $X^{(n)}$ does not have a $\mathbf{d}^{(2n-1)}$-computable compact presentation.*

3 Degree spectra of Polish spaces

3.1 Basic tools

3.1.1 Trees of connected components

Given a compact presentation of X, we will construct a finitely branching tree T_X of components and a linear order on components of X. A similar notion has been studied in Brattka et al. [5].

Recall from Section 2.2 that a compact presentation of X yields a strong refinement sequence $(\mathcal{U}_s)_s$ of open covers whose members are rational open in the Hilbert cube Q, and then $X_s = \bigcup \mathcal{U}_s$ provides a rational open approximation of X in Q. As disjointness of rational open sets in Q is decidable, one can effectively decompose X_s into finitely many connected components $\{C_0^s, \dots, C_{\ell(s)}^s\}$. Then, we get the *tree of components*, T_X, which consists of sequences $(C_{u(0)}^0, C_{u(1)}^1, \dots, C_{u(k)}^k)$ of connected components such that $C_{u(i+1)}^{i+1} \subseteq C_{u(i)}^i$ for each $i < k$. Note that a node of T_X can also be considered as a refinement sequence $(\mathcal{V}_0, \dots, \mathcal{V}_k)$ such that $\mathcal{V}_s \subseteq \mathcal{U}_s$ and $\bigcup \mathcal{V}_s$ is a component of X_s.

Then, each infinite path through T_X corresponds to a connected component of X. Note that the construction of T_X from a compact presentation of X is effective. If C is a component of X_s, then there is a unique node $\sigma_C \in T_X$ of length $s+1$ such that the last entry of σ_C is C. We label each component C by the least index i such that $a_i \in C$, where recall that $(a_i)_i$ is a dense sequence of X. Then, we order the components of X_s as follows: For components C and D of X_s, define $C < D$ if either the last branching height of σ_C in T_X is smaller than that of σ_D or the branching heights are the same and the label of C is smaller than that of D. Roughly speaking, $C < D$ iff either C stabilizes earlier than D, or else C and D stabilize at the same stage, and C contains a smaller indexed point than D. One can also order all infinite paths through T_X by a similar argument.

3.1.2 Learning the dimension of a sphere

Consider the following situation: We are informed that, for some d, the d-dimensional sphere $\mathbf{S}^d$ is embedded into the Hilbert cube Q, and moreover, a compact presentation of the embedded image is given to us, but we do not know the dimension d. How can we guess the correct dimension d?

First, given a finite open covering $\mathcal{U}_s$ of a compact set $X \subseteq Q$, let us consider the *formal nerve* $\mathcal{N}_s$ of $\mathcal{U}_s$, which is an abstract simplicial complex defined as follows: A finite set $J \subseteq \ell(s)$ belongs to $\mathcal{N}_s$ if and only if the formal intersection of $\{B(a_n; 2^{-s}) : n \in J\}$ is nonempty. As the formal nerve of a given covering of X is a finite abstract simplicial complex, one can compute the homology groups of the nerve. However, a compact presentation of X may give an extremely wild embedding of X into the Hilbert cube. For instance, the embedded image of X in the Hilbert cube may look like the Alexander horned sphere (which is homeomorphic to $\mathbf{S}^2$). In general, the homology groups of a compact space can be completely different from the homology groups of its finitary approximations. For instance, consider the Warsaw circle, which is approximated by a sequence of 1-spheres (i.e., circles). Thus, it is not so obvious whether one can eventually recognize the correct dimension from information of homology groups of finitary approximations of X, so we take a slightly different approach.

Let us first describe a geometric idea behind our algorithm Ψ guessing the dimension of a sphere. If $X \simeq \mathbf{S}^d$, for any $e \neq d$, any occurrence of an e-dimensional hole (i.e., an e-cycle which is not an e-boundary) will be eventually broken, and only tiny or thin e-dimensional holes may be detected at later stages. On the other hand, we will eventually detect a (non-thin portion of) d-dimensional hole which survives forever. Hence, our learning algorithm Ψ just returns the dimension of a longest-surviving portion of a hole at each stage.

For a given d, we now focus on d-dimensional holes (that look like d-spheres) in $\mathcal{U}_s$ and $\mathcal{N}_s$ (or their finite modifications). Formally speaking, a *d-dimensional hole* or simply a *d-hole* in the nerve $\mathcal{N}_s$ is a d-dimensional cycle which is not a d-dimensional boundary in $\mathcal{N}_s$. A face α in the nerve $\mathcal{N}_s$ is associated with a finite sequence $\mathcal{U}_s^\alpha$ from $\mathcal{U}_s$. The collection $\mathcal{U}_s^\alpha$ or its intersection is essentially a geometric realization of α, so we call it a *formal geometric realization of* α. In a similar manner, one can also consider a formal geometric realization $\mathcal{U}_s^\gamma$ of a chain γ in the nerve $\mathcal{N}_s$. More formally, $\mathcal{U}_s^\gamma$ is the union of formal geometric realizations of all faces contained in γ. Note that if γ is a d-hole, even if we delete duplicated entries in $\mathcal{U}_s^\gamma$, the resulting set still determines the same formal geometric d-hole in X_s. Thus, one can assume that a d-hole contains no repetition of faces. In particular, one can compute a canonical index of the finite set of all d-holes in the nerve $\mathcal{N}_s$.

Now, we say that γ is a *d-hole detected at s* if it is a d-hole in the induced nerve $\mathcal{N}_s$ of a finite modification of $\mathcal{U}_s$. For $t > s$, we say that such a d-hole γ *survives at t* if, for any face α in γ, the formal geometric realization $\bigcap \mathcal{U}_s^\alpha$ is nonempty and connected in X_t. Note that one can effectively check whether a given d-hole survives at t or not.

Lemma 3.1. *Assume that a compact presentation of a topological sphere X is given (i.e., $X \simeq \mathbf{S}^d$ for some $d \in \omega$). Then $X \simeq \mathbf{S}^d$ if and only if there is a d-hole which is detected at some stage, and survives forever.*

Proof. We first show the forward direction, so assume $X \simeq \mathbf{S}^d$. For a sufficiently large stage s, $\mathcal{N}_s$ must contain a *true* d-hole γ_s: We first consider the d-sphere $\mathbf{S}^d$ as a subspace of the Hilbert cube Q. Let $h\colon Q \to Q$ be a continuous extension of a homeomorphism $X \simeq \mathbf{S}^d$. Note that such an h exists since X is a closed subset of Q and the Hilbert cube Q is an absolute extensor (by the coordinate-wise application of the Tietze extension theorem). Without loss of generality, one can assume that $x \notin X$ implies $h(x) \notin \mathbf{S}^d$: This is because, let us consider $\{0\} \times \mathbf{S}^d = \{(0, x_0, x_1, \dots) \in Q : (x_0, x_1, \dots) \in \mathbf{S}^d\}$ instead of $\mathbf{S}^d$. Then, as above, we have a continuous function $h\colon Q \to Q$ such that $h \upharpoonright X$ is a homeomorphism between X and $\{0\} \times \mathbf{S}^d$. Since Q is perfectly normal and X is closed, there is a continuous function $g\colon Q \to [0,1]$ such that $g(x) = 0$ if and only if $x \in X$. Now, replace the first coordinate of $h(x)$ with $g(x)$. This fulfills the desired condition.

Let $\mathcal{B}$ be a collection of basic open balls in the Hilbert cube Q such that $\{B \cap \mathbf{S}^d : B \in \mathcal{B}\}$ is a cover of $\mathbf{S}^d$ of order $d+1$, whose nerve gives a rational triangulation of $\mathbf{S}^d$. Then consider $h^*\mathcal{B} = \{h^{-1}[B] : B \in \mathcal{B}\}$, which is a finite open cover of X, and note that $h^{-1}[B \cap \mathbf{S}^d] = h^{-1}[B] \cap h^{-1}[\mathbf{S}^d] = h^{-1}[B] \cap X$, which is homeomorphic to the connected set $B \cap \mathbf{S}^d$. Then, let δ be a Lebesgue number of $h^*\mathcal{B}$, that is, if an open cover $\mathcal{U}$ of X has mesh less than δ then any $U \in \mathcal{U}$ is a subset of $h^{-1}[B]$ for some $B \in \mathcal{B}$. For any sufficiently large s, the mesh of $\mathcal{U}_s$ is less than δ, and thus, $\mathcal{U}_s$ is a refinement of $h^*\mathcal{B}$. For each $B \in \mathcal{B}$, put $U_B = \bigcup\{U \in \mathcal{U}_s : U \subseteq h^{-1}[B]\}$, and define $\mathcal{U}_s^* = \{U_B : B \in \mathcal{B}\}$. Then, $\mathcal{U}_s^*$ is a finite modification of $\mathcal{U}_s$. Now one can obtain a minimal d-hole by listing collections of members from $\mathcal{U}_s^*$ whose intersection is nonempty and connected in X (not just in X_s), and we call it a true d-hole. Although this procedure may be not effective, we do not need effectivity. Then, clearly, such a true d-hole survives at any stage $t > s$.

We next show the converse direction; that is, if $X \not\simeq \mathbf{S}^d$, then every detected d-holes does not survive at some later stage. Let γ be a d-hole in $\mathcal{N}_s$, which yields a geometric d-hole $\mathcal{U}_s^\gamma$ of $\mathcal{U}_s$. As the shape of any subspace of X is different from $\mathbf{S}^d$, if d is bigger than the dimension of X, then such a hole will be eventually broken at some stage by compactness. If d is smaller than the dimension of X, then a d-cycle γ' can be described on the surface of X, but such a γ' must be a boundary, so in particular, γ' is not a d-hole. As $X \not\simeq \mathbf{S}^d$, this observation implies that the geometric d-hole $\mathcal{U}_s^\gamma$ must add a new route between two disjoint locations (which may look like a wormhole) on the surface of the d-sphere X. But this route does not exist in X, and moreover, any two disjoint locations on the surface of a sphere must be separated; hence, by compactness, all such routes are broken at some stage.

Formally speaking, in this case, there is a face α in γ whose geometric realization $\bigcap \mathcal{U}_s^\alpha$ (which corresponds to a part of a new route in the above sense) is either empty or disconnected in X, and then it is detected at some stage $t \geq s$ by compactness. This means that any d-hole detected at s does not survive at any stage after t. ☐

Lemma 3.2. *There exists a limit computable function which, given a compact presentation of X, returns $d \in \omega$ such that $X \simeq \mathbf{S}^d$ whenever such a d exists.*

Proof. At each stage s, our algorithm Ψ lists all holes detected at some stage $t \leq s$, and check if a given such a hole still survives at s. Then Ψ returns the least dimension n of a longest-surviving hole at s; that is, first compute the least $t \leq s$ such that there is a hole γ detected at t and survives at s, and then the least dimension n of such a hole. This leaning algorithm Ψ converges to the correct dimension d: By (the forward direction in) Lemma 3.1, there is a d-hole detected at some t which survives forever. Similarly, by (the backward direction in) Lemma 3.1, if $e \neq d$, then any e-hole detected at some $u \leq t$ does not survive at some later stage. By finiteness of nerves, there are only finitely many holes detected at some $u \leq t$. Thus, there is s such that any e-hole detected at some $u \leq t$ does not survive at s. This means that a d-hole eventually becomes a longest-surviving hole; hence Ψ returns d at any stages after s. □

3.2 Controlling the single jump

Theorem 3.3. *For any degree* $\mathbf{d}$*, there is a compact metrizable space* $\mathcal{Z}_{\mathbf{d}}$ *whose compact degree spectrum is* $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}'\}$.

Proof. For $D \in \mathbf{d}$, define $\mathcal{Z}_{\mathbf{d}}$ as the one-point compactification of the following locally compact space:

$$\coprod_{n \in D} \mathbf{S}^{2n+1} \amalg \coprod_{n \notin D} \mathbf{S}^{2n+2},$$

where $\mathbf{S}^d$ is the d-dimensional sphere. Assume that an $\mathbf{x}$-computable presentation of $\mathcal{Z}_{\mathbf{d}}$ is given. Then, one can effectively construct the tree T_Z of components of $\mathcal{Z}_d$ as in Section 3.1.1.

We use the learning algorithm Ψ in Lemma 3.2 to find a component of $\mathcal{Z}_{\mathbf{d}}$ whose dimension is n. Note that there is unique such a component. At stage s, look at each component C of the height s in the tree T_Z, and then apply the learning algorithm Ψ in Lemma 3.2 to the component C. We label each node C of the height s in the tree T_Z by the current guess n of Ψ; that is, Ψ currently considers that C looks like the n-sphere $\mathbf{S}^n$. At stage s, for any $n \leq s$, we check if there is a connected component of height s labeled by either $2n+1$ or $2n+2$. If there are such components, we follow the label of the $<$-least component C of height s. If C is labeled by $2n+1$ then we guess $n \in D$, and if C is labeled by $2n+2$ then we guess $n \notin D$, at stage s. If there is no such component, then we guess $n \in D$ at stage s.

We claim that this algorithm eventually produces the correct guess. Now, note that $\mathcal{Z}_{\mathbf{d}}$ has exactly one copy of $\mathbf{S}^{2n+1}$ or $\mathbf{S}^{2n+2}$. Let C be a connected component in $\mathcal{Z}_{\mathbf{d}}$ corresponding to such a copy. First note that C has no point which is the limit of other components, and therefore, by compactness, C corresponds to an isolated path p_C in T_Z (i.e., for almost all s, the cover $\mathcal{U}_s$ distinguishes C from other components). Hence, there are only finitely many infinite paths through T_Z which is $<$-smaller than p_C. By Lemma 3.2, the label of $p_C \upharpoonright s$ converges to the correct dimension $2n+1$ or $2n+2$. As C is the unique component of dimension n, for any path $p < p_C$ (which is always isolated in T_Z), the label of $p \upharpoonright s$ converges to some value different from $2n+1$ and $2n+2$. Hence, for any sufficiently large s, $p_C \upharpoonright s$ must be the $<$-least one whose label is either $2n+1$ or $2n+2$. This means that our algorithm eventually follows this correct component, and returns the correct guess. In other words, this algorithm is an $\mathbf{x}$-computable approximation procedure which decides the value of D in the limit. Consequently, we obtain $\mathbf{d} \leq \mathbf{x}'$.

Conversely, assume $\mathbf{d} \leq_T \mathbf{x}'$. Then, fix an $\mathbf{x}$-computable approximation procedure φ

converging to $D \in \mathbf{d}$; that is, $D(n) = \lim_s \varphi(n, s)$. We will construct a presentation of a space $\mathcal{Z}$. First enumerate a point z in $\mathcal{Z}$, and then prepare for infinitely many pairwise separated regions $(R_i)_{i\in\omega}$, where R_i is 2^{-i}-close to z. Inside the region R_d, we will construct either $\mathbf{S}^{2d+1}$ or $\mathbf{S}^{2d+2}$. First note that, for any ε, every $(2d+1)$-sphere $\mathbf{S}^{2d+1}$ is ε-close to a $(2d+2)$-sphere $\mathbf{S}^{2d+2}$ (w.r.t. the Hausdorff distance) and vice versa. At stage s, if we see that $\varphi(n, s) = 1$, then, inside R_d, we start to construct $\mathbf{S}^{2d+1}$ which is 2^{-s}-close to the one constructed at the previous stage. Similarly, if we see that $\varphi(n, s) = 1$, then, inside R_d, we start to construct $\mathbf{S}^{2d+2}$ which is 2^{-s}-close to the one constructed at the previous stage. As $\varphi(n, s)$ stabilizes to the correct value $D(n)$ after some stage s, we eventually construct $\mathbf{S}^{2d+1}$ if $n \in D$, and $\mathbf{S}^{2d+2}$ if $n \notin D$. This construction clearly produces a space $\mathcal{Z}$ which is homeomorphic to $\mathcal{Z}_{\mathbf{d}}$. Moreover, our construction is $\mathbf{x}$-computable, and indeed, it is not hard to make this construction $\mathbf{x}$-effectively compact. Hence, $\mathcal{Z}_{\mathbf{d}}$ has an $\mathbf{x}$-computable compact presentation. □

Theorem 3.4. *For any degree $\mathbf{d}$, there is a compact metrizable space $\mathcal{X}_{\mathbf{d}}$ whose Polish degree spectrum is $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}''\}$, but compact degree spectrum is $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}'\}$.*

It suffices to show the following:

Lemma 3.5. *For any $D \subseteq \omega$, there is a compact metrizable space $\mathcal{X}_D$ such that for any $Z \subseteq \omega$,*

$$D \text{ is } \Sigma^0_2 \text{ relative to } Z \iff \mathcal{X}_D \text{ has a } Z\text{-computable compact presentation.}$$
$$D \text{ is } \Sigma^0_3 \text{ relative to } Z \iff \mathcal{X}_D \text{ has a } Z\text{-computable Polish presentation.}$$

Proof. We denote by $\tilde{\mathbf{S}}^n$ the topological sum of countably many copies of the n-sphere $\mathbf{S}^n$. Then, let $\mathcal{X}_D$ be the one-point compactification of the following locally compact space:

$$\coprod_{n\in D} \tilde{\mathbf{S}}^{n+1} \amalg A,$$

where A is a countable set of isolated points. The same argument as in the proof of Theorem 3.3 shows that if $\mathcal{X}_D$ has a Z-computable compact presentation then D is Σ^0_2 relative to Z: $n \in D$ if and only if there is an isolated path through the Z-computable tree of components induced from $\mathcal{X}_D$ such that some $(n+1)$-hole survives forever along this path. This is clearly a $\Sigma^0_2(Z)$ condition. Now, assume that $\mathcal{X}_D$ has a Z-computable Polish presentation. Then, by Lemma 2.1, $\mathcal{X}_D$ has a Z'-computable compact presentation. Therefore, D is Σ^0_2 relative to Z'; that is, D is Σ^0_3 relative to Z.

Conversely, assume that D is Σ^0_3 relative to Z. Therefore, there is a Z-computable set $S \subseteq \omega$ such that $n \in D$ if and only if there is $a \in \omega$ such that $(n, a, b) \in S$ for infinitely many $b \in \omega$. We will construct a presentation of a space $\mathcal{X}$. First enumerate a point z in $\mathcal{X}$, and then prepare for infinitely many pairwise separated regions $(R_i)_{i\in\omega}$, where R_i is 2^{-i}-close to z. For any $n, a, k \in \omega$, inside the region $R_{\langle n,a,k\rangle}$, we will construct either a copy of $\mathbf{S}^{n+1}$ or a finite set of isolated points. Fix a computable homeomorphic copy $S_{n,a,k}$ of $\mathbf{S}^{n+1}$ inside $R_{\langle n,a,k\rangle}$, but note that we do not enumerate $S_{n,a,k}$ into $\mathcal{X}$. Let $(q^{n,a,k}_b)_{b\in\omega}$ be a computable dense sequence in $S_{n,a,k}$. Then, we put

$$A_{n,a,k} = \{q^{n,a,k}_b : (\exists s > b)\ (n, a, k, s) \in S\}.$$

Clearly, $A^{n,a,k}$ is c.e. relative to Z. Then we take a Z-computable enumeration of $\bigcup_{n,a,k} A_{n,a,k}$ as a dense sequence of our space $\mathcal{X}$.

It is not hard to verify that, if $n \in D$, then, by our choice of S, there is a such that, for any $k \in \omega$, the completion of $A_{n,a,k}$ is homeomorphic to $S_{n,a,k} \simeq \mathbf{S}^{n+1}$. In particular, $\mathcal{X}$ contains countably many copies of $\mathbf{S}^{n+1}$. If $n \notin D$, then for any a and k, the set $A_{n,a,k}$ is a finite set of isolated points. In particular, $\mathcal{X}$ contains no copy of $\mathbf{S}^{n+1}$. If necessary, one can always add a countable set of isolated points. Consequently, $\mathcal{X}$ is homeomorphic to $\mathcal{X}_D$. Moreover, our construction is Z-computable, and hence, $\mathcal{X}_D$ has a Z-computable Polish presentation. $\square$

Proof of Theorem 3.4. Given $D \subseteq \omega$ of Turing degree $\mathbf{d}$, consider $\mathcal{X}_{\mathbf{d}} = \mathcal{X}_D \amalg \mathcal{X}_{D^c}$. Then, by Lemma 3.5, $\mathcal{X}_{\mathbf{d}}$ satisfies the desired condition. $\square$

Remark 3.6. One can modify the proof of Theorem 3.4 to ensure that $\mathcal{X}_D$ is perfect, by using the one-point compactification of the following:

$$\coprod_{n \in D} \tilde{\mathbf{S}}^{n+1} \amalg \coprod_{n \in \omega} \tilde{\mathbf{I}}^n,$$

where $\tilde{\mathbf{I}}^n$ is the topological sum of countably many copies of the n-dimensional cube $[0,1]^n$. As $[0,1]^n$ has no hole, the argument in Theorem 3.3 still works. For the other direction, replace a finite set of isolated points in the proof of Theorem 3.4 with a finite set of pairwise disjoint small segments (each of which is homeomorphic to $[0,1]^n$) of the surface of $\mathbf{S}^n$, and put countably many copies of $[0,1]^n$ at somewhere else.

We now solve Question 2 affirmatively.

Corollary 3.7. *There exist a low$_3$ Turing degree $\mathbf{d} \leq \mathbf{0}'$ and a perfect Polish space X such that X is $\mathbf{d}$-computably presentable, but not computably presentable.*

Proof. Let $\mathbf{d} \leq \mathbf{0}'$ be a low$_3$ Turing degree which is not low$_2$, i.e., $\mathbf{d}'' \not\leq \mathbf{0}''$. The Polish degree spectrum of the space $\mathcal{X}_{\mathbf{d}''}$ in Theorem 3.4 is $\{\mathbf{x} : \mathbf{d}'' \leq \mathbf{x}''\}$, which contains $\mathbf{d}$, but does not contain $\mathbf{0}$ since $\mathbf{d}$ is not low$_2$. In other words, the space $\mathcal{X}_{\mathbf{0}'''}$ is $\mathbf{d}$-computably presentable, but not computably presentable. $\square$

We also note that $\mathcal{X} = \mathcal{X}_{\mathbf{0}'''}$ is a Polish space such that

$$\mathbf{d} \text{ is high}_2 \iff \mathcal{X} \text{ has a } \mathbf{d}\text{-computable presentation.}$$

Corollary 3.8. *There is a compact perfect Polish space $\mathcal{X}$ which has a computable presentation, but has no presentation which makes $\mathcal{X}$ computably compact.*

Proof. The Polish degree spectrum of the space $\mathcal{X}_{\mathbf{0}''}$ in Theorem 3.4 is $\{\mathbf{x} : \mathbf{0}'' \leq \mathbf{x}''\}$, which contains all degrees, while the compact degree spectrum of $\mathcal{X}_{\mathbf{0}''}$ is $\{\mathbf{x} : \mathbf{0}'' \leq \mathbf{x}'\}$, which does not contain $\mathbf{0}$. In other words, the space $\mathcal{X}_{\mathbf{0}''}$ has a computable presentation, but has no presentation which makes $\mathcal{X}_{\mathbf{0}''}$ effectively compact. $\square$

3.3 The iterated jumps in compact degree spectra

In this section, we combine higher-dimensional spheres (Sections 3.1.2 and 3.2) and countable topological spaces (Section 2.3) to control the iterated jumps.

Theorem 3.9. *For any degree* $\mathbf{d}$*, there is a compact metrizable space* $\mathcal{Z}_{\mathbf{d},2}$ *whose compact degree spectrum is* $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}'''\}$.

It suffices to show the following:

Lemma 3.10. *For any* $D \subseteq \omega$*, there is a compact metrizable space* $\mathcal{Z}_{D,2}$ *such that for any* $Z \subseteq \omega$,

$$D \text{ is } \Sigma^0_4 \text{ relative to } Z \iff \mathcal{Z}_{D,2} \text{ has a } Z\text{-computable compact presentation.}$$

Proof. Let z be a unique point in the space $\omega^\alpha + 1$ whose Cantor-Bendixson rank is α, and let p be any point in the n-sphere $\mathbf{S}^n$. Then, consider the *wedge sum* of pointed spaces $(\mathbf{S}^n, p)$ and $(\omega^\alpha + 1, z)$. In other words, we combine two spaces $\mathbf{S}^n$ and $\omega^\alpha + 1$ by gluing p and z. More explicitly, let us consider the equivalence relation $\sim$ with $p \sim z$, and take the quotient $(\mathbf{S}^n \cup (\omega^\alpha + 1))/\sim$. We denote the resulting quotient space by $\mathbf{S}^n \vee (\omega^\alpha + 1)$. Then, we define

$$\mathcal{S}^{n+1}_D = \begin{cases} \mathbf{S}^{n+1} \vee (\omega + 1) & \text{if } n \in D, \\ \mathbf{S}^{n+1} \vee (\omega^2 + 1) & \text{if } n \notin D, \end{cases}$$

Then, let $\mathcal{Z}_{\mathbf{d},2}$ be the one-point compactification of the following locally compact space:

$$\coprod_{n \in \omega} \mathcal{S}^{n+1}_D \amalg (\omega^2 + 1).$$

Here we note that $\omega^2 + 1$ contains countably many isolated points, and countably many copies of $\omega + 1$.

Given a Z-computable compact presentation of $\mathcal{Z}_{D,2}$, let us first consider the Z-computable tree T_Z of components of $\mathcal{Z}_{D,2}$ (see Section 3.1.1). Recall that a component of $\mathcal{Z}_{D,2}$ is an infinite path through T_Z. There is a unique component in $\mathcal{Z}_{D,2}$ which is homeomorphic to $\mathbf{S}^{n+1}$, and let α_{n+1} be the corresponding infinite path through T_Z. At each stage s, compute a node η_s of T_Z which has a longest-surviving $(n+1)$-hole among nodes of T_Z of length s. Then, η_s is an initial segment of α_{n+1} for almost all s.

The key observation is that if $n \notin D$, then $\mathcal{S}^{n+1}_D \simeq \mathbf{S}^{n+1} \vee (\omega^2 + 1)$, so any neighborhood of the $(n+1)$-sphere $\mathbf{S}^{n+1}$ has a copy of $\omega + 1$ which is separated from the sphere. This means that the Cantor-Bendixson rank $\rho(\alpha_{n+1})$ of α_{n+1} in the space $[T_Z]$ is 2. On the other hand, in the case $n \in D$, we have $\mathcal{S}^{n+1}_D \simeq \mathbf{S}^{n+1} \vee (\omega + 1)$, so if a neighborhood of the $(n+1)$-sphere $\mathbf{S}^{n+1}$ is sufficiently small, then it only contains isolated points except for the sphere itself. This means that the Cantor-Bendixson rank $\rho(\alpha_{n+1})$ of α_{n+1} in the space $[T_Z]$ is 1. By this observation, $n \notin D$ if and only if $\rho(\alpha_{n+1}) \geq 2$. In other words, for any s there are $t > s$ and $\sigma \in T_Z$ such that $\sigma \succeq \eta_t$, $\sigma \not\succeq \eta_{t+1}$, and σ has some extension of rank 1 in $[T_Z]$; that is, for any ℓ there is a branching node $\tau \in T_Z$ of length ℓ extending σ. This is clearly a $\Pi^0_4(Z)$ property.

For the other direction, assume that D is Σ^0_4 relative to Z. Then, there is a computable set $S \subseteq \omega^2$ such that $n \notin D$ if and only if $\exists^\infty a \exists^\infty b\,(n,a,b) \in S$; in other words, infinitely many sections $S^a = \{b : (n,a,b) \in S\}$ of S have infinitely many elements. We will construct a presentation of a space $\mathcal{Z}$. First enumerate a point z and a copy of ω^2+1 into $\mathcal{Z}$, and then we prepare for infinitely many pairwise separated regions $(R_i)_{i\in\omega}$, where R_i is 2^{-i}-close to z. Inside R_n, we start by describing $\mathbf{S}^{n+1} \vee (\omega+1)$. Let $\{p_a, p_\infty\}_{a\in\omega}$ be the copy of $\omega+1$ in this space, and note that p_∞ also belongs to $\mathbf{S}^{n+1}$. For each $a \in \omega$, we consider an (imaginary) sequence $\{p_{ab}\}_{b\in\omega}$ converging to p_a, and for any f, $\{p_{a,f(a)}\}_{a\in\omega}$ converges to p_∞. For instance, consider a triangle area, one of whose vertices corresponds to p_∞; arrange $\{p_a\}_{a\in\omega}$ on a side of the triangle, and put $\{p_{ab}\}_{a,b\in\omega}$ inside the triangle area. Then, we only enumerate $\{p_{ab} : (n,a,b) \in S\}$ into the region R_n.

If $(n,a,b) \in S$ for infinitely many b, then p_a is a rank 1 point; otherwise, p_a is isolated. Hence, if $\exists^\infty a \exists^\infty b\,(n,a,b) \in S$ is true (i.e., $n \notin D$), there are infinitely many a such that p_a is of rank 1, so p_∞ is the limit of rank 1 points. In this case, this space restricted to the region R_n is homeomorphic to $\mathbf{S}^{n+1} \vee (\omega^2+1)$. If $\exists^\infty a \exists^\infty b\,(n,a,b) \in S$ is false (i.e., $n \in D$), p_a is isolated for all but finitely many a. Hence, p_∞ is the limit of rank 0 points, but not the limit of rank 1 points. There may be other rank 1 points, but they are separated from the sphere $\mathbf{S}^{n+1}$. Hence, our space restricted to the region R_n is homeomorphic to the separated union of $\mathbf{S}^{n+1} \vee (\omega+1)$ and at most finitely many copies of $\omega+1$. Up to homeomorphism, the latter part is absorbed into a copy of ω^2+1 in some other region. Consequently, $\mathcal{Z}$ is homeomorphic to $\mathcal{Z}_{D,2}$. Moreover, our construction is Z-computable, and hence $\mathcal{Z}_{D,2}$ has a Z-computable compact presentation. □

Theorem 3.11. *For any degree $\mathbf{d}$, there is a compact metrizable space $\mathcal{Z}_{\mathbf{d},3}$ whose compact degree spectrum is $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}^{(5)}\}$.*

It suffices to show the following:

Lemma 3.12. *For any $D \subseteq \omega$, there is a compact metrizable space $\mathcal{Z}_{D,3}$ such that for any $Z \subseteq \omega$,*

$$D \text{ is } \Sigma^0_6 \text{ relative to } Z \iff \mathcal{Z}_{D,3} \text{ has a } Z\text{-computable compact presentation.}$$

Proof. Given D, let us consider:

$$\mathcal{S}^{n+1}_{D,k} = \begin{cases} \mathbf{S}^{n+1} \vee (\omega^{k-1}+1) & \text{if } n \in D, \\ \mathbf{S}^{n+1} \vee (\omega^k+1) & \text{if } n \notin D, \end{cases}$$

Then, let $\mathcal{Z}_{D,3}$ be the one-point compactification of the following locally compact space:

$$\coprod_{n\in\omega} \mathcal{S}^{n+1}_{D,3} \amalg (\omega^3+1).$$

It is not hard to see that the Cantor-Bendixson derivative of $\mathcal{Z}_{D,3}$ is homeomorphic to $\mathcal{Z}_{D,2}$ in Theorem 3.9. Assume that $\mathcal{Z}_{D,3}$ has a Z-computable presentation. Then, by Lemma 2.5, $\mathcal{Z}_{D,2}$ has an Z''-computable presentation. By Theorem 3.9, we have D is Σ^0_4 relative to Z''; hence, Σ^0_6 relative to Z.

For the other direction, assume that D is Σ^0_6 relative to Z. Then, there is a computable set $S \subseteq \omega^2$ such that $n \notin D$ if and only if $\exists^\infty a \exists^\infty b \exists^\infty c\,(n,a,b,c) \in S$. We will construct a presentation of a space $\mathcal{Z}$. First enumerate a point z and a copy of ω^3+1 into $\mathcal{Z}$, and then we prepare for infinitely many pairwise separated regions $(R_i)_{i\in\omega}$, where R_i is 2^{-i}-close to z. Inside R_n, let $\{p_a, p_\infty\}_{a\in\omega}$ be the copy of $\omega+1$ in this space such that p_∞ is a point in the surface of a copy S of the $(n+1)$-sphere $\mathbf{S}^{n+1}$. For each $a \in \omega$, we consider an (imaginary) sequence $P_a = \{p_{ab}\}_{b\in\omega}$ converging to p_a, and the diameter of P_a is at most 2^{-a}. Similarly, consider $P_{ab} = \{p_{abc}\}_{c\in\omega}$ converging to p_{ab}, and the diameter of P_{ab} is at most 2^{-a-b}. Then, we enumerate S, $\{p_a, p_\infty\}_{a\in\omega}$, $\{p_{ab}\}_{a,b\in\omega}$, and $\{p_{abc} : (n,a,b,c) \in S\}$ into the region R_n.

If $(n,a,b,c) \in S$ for infinitely many c, then p_{ab} is a rank 1 point; otherwise, p_{ab} is isolated. Hence, if $\exists^\infty b \exists^\infty c\,(n,a,b,c) \in S$ is true, there are infinitely many b such that p_{ab} is of rank 1, so p_a is of rank 2. Otherwise, for almost all b, p_{ab} is isolated, so p_a is of rank 1. If $\exists^\infty a \exists^\infty b \exists^\infty c\,(n,a,b,c) \in S$ is true (i.e., $n \notin D$), there are infinitely many a such that p_a is of rank 2, so p_∞ is the limit of rank 2 points. In this case, this space restricted to the region R_n is homeomorphic to $\mathbf{S}^{n+1} \vee (\omega^3+1)$. If $\exists^\infty a \exists^\infty b \exists^\infty c\,(n,a,b,c) \in S$ is false (i.e., $n \in D$), p_a is of rank 1 for almost all a. Hence, p_∞ is the limit of rank 1 points, but not the limit of rank 2 points. There may be other rank 2 points, but they are separated from the sphere $\mathbf{S}^{n+1}$. Hence, our space restricted to the region R_n is homeomorphic to the separated union of $\mathbf{S}^{n+1} \vee (\omega^2+1)$ and at most finitely many copies of ω^2+1. Up to homeomorphism, the latter part is absorbed into a copy of ω^3+1 in some other region. Consequently, $\mathcal{Z}$ is homeomorphic to $\mathcal{Z}_{D,3}$. Moreover, our construction is Z-computable, and hence $\mathcal{Z}_{D,3}$ has a Z-computable compact presentation. □

Theorem 3.13. *For any degree $\mathbf{d}$ and $n \in \omega$, there is a compact metrizable space $\mathcal{Z}_{\mathbf{d},n}$ whose compact degree spectrum is $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}^{(2n-1)}\}$.*

It suffices to show the following:

Lemma 3.14. *For any $D \subseteq \omega$ and $n \in \omega$, there is a compact metrizable space $\mathcal{Z}_{D,n}$ such that for any $Z \subseteq \omega$,*

$$D \text{ is } \Sigma^0_{2n} \text{ relative to } Z \iff \mathcal{Z}_{D,n} \text{ has a } Z\text{-computable compact presentation.}$$

Proof. Given $D \subseteq \omega$, let $\mathcal{Z}_{D,n}$ be the one-point compactification of the following locally compact space:

$$\coprod_{d\in\omega} \mathcal{S}^{d+1}_{D,n} \amalg (\omega^n + 1),$$

where $\mathcal{S}^{d+1}_{D,n}$ is the space defined in the proof of Theorem 3.11. Then, the Cantor-Bendixson derivative of $\mathcal{Z}_{D,n}$ is homeomorphic to $\mathcal{Z}_{D,n-1}$. The remaining argument is the same as in the proof of Theorem 3.11. □

Recall that we concluded from Theorem 2.11 that Lemma 2.5 is almost optimal. As a corollary of Theorem 3.13, one can conclude a similar, but slightly different result. Recall from Theorem 2.9 that the compact degree spectra $Sp_c(\mathbf{CP}_0)$ of zero-dimensional compacta is equal to the degree spectra $Sp(\mathbf{BA})$ of Boolean algebras.

For a special property of $Sp(\mathbf{BA})$, Jockusch-Soare [15] showed that if the isomorphism type of a Boolean algebra can have nth jump degree for some $n \in \omega$ then it already has a computable presentation. Here, (the isomorphism type of) a structure X has nth jump degree if there is a least Turing degree $\mathbf{d}$ such that the nth jump of a presentation of X bounds $\mathbf{d}$. If such a $\mathbf{d}$ is nonzero, we say that X has nontrivial nth jump degree.

By the above observation, no homeomorphism type of a zero-dimensional compactum has nontrivial nth jump degree for any $n \in \omega$. On the other hand, Theorem 3.3 shows the existence of an infinite dimensional compactum which has nontrivial first jump degree. We now combine these observations with Theorem 3.13.

Corollary 3.15. *There is a compact metrizable space $\mathcal{X}$ which has a computably compact presentation, but its nth Cantor-Bendixson derivative $\mathcal{X}^n$ has no $\mathbf{0}^{(2n-1)}$-computably compact presentation, but the homeomorphism type of $\mathcal{X}^n$ has 3rd jump degree.*

Proof. Let us consider the space $\mathcal{X} = \mathcal{Z}_{\mathbf{d},n+2}$ in Theorem 3.13. Then, the compact spectrum of $\mathcal{X}$ is $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}^{(2n+3)}\}$. On the other hand, the compact spectrum of its nth derivative $\mathcal{X}^n \simeq \mathcal{Z}_{\mathbf{d},2}$ is $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}'''\}$. If we take $\mathbf{d} = \mathbf{0}^{(2n+3)}$ then $\mathcal{X}$ satisfies the desired condition. □

Recall that a Turing degree $\mathbf{x}$ is high$_n$ if $\mathbf{0}^{(n+1)} \leq \mathbf{x}^{(n)}$. Theorem 3.13 (with $\mathbf{d} = \mathbf{0}^{(2n)}$) shows that the class of high$_{2n-1}$-degrees is a compact degree spectrum of a compact metrizable space.

3.4 The iterated jumps in Polish degree spectra

By using the idea in the previous section, we show an analogue of Theorem 3.13 for Polish degree spectra.

Let $\coprod_\omega X$ be the separated union of countably many copies of X, and $\alpha_\omega X$ be its one-point compactification, that is, $\alpha_\omega X$ denotes the one-point compactification of the separated union of countably many copies of X. When considering the wedge sum, we often think of $\alpha_\omega X$ as a pointed space $(\alpha_\omega X; \infty)$ whose basepoint is ∞, the point at infinity. Define α_ω^ξ as the ξth iteration of the compactification α_ω. For instance, $\alpha_\omega 1 \simeq \omega + 1$ with the basepoint ω, and $\alpha_\omega^{\xi+1} 1 \simeq \alpha_\omega(\omega^\xi + 1) \simeq \omega^{\xi+1} + 1$ with the basepoint $\omega^{\xi+1}$ (a unique rank $\xi + 1$ point).

For a connected space S, we define the *S-rank* of a component C of X as follows: The S-rank of C is > 0 if for any neighborhood N of C there is an other component in N which is homeomorphic to S. The S-rank of x is $> \alpha$ if for any neighborhood N of C there is an other component in N of rank $\geq \alpha$. The S-rank of X is the supremum of the S-ranks of components of X.

We inductively define the following compact spaces:

$$\mathbf{S}\Sigma_1 = \mathbf{S}\Sigma_1^+ = 1, \quad \mathbf{S}\Sigma_{2n+3} = \alpha_\omega \mathbf{S}\Sigma_{2n+1}^+, \quad \mathbf{S}\Sigma_{2n+3}^+ = \mathbf{S}\Sigma_{2n+3} \amalg \mathbf{S}\Pi_{2n+1},$$

$$\mathbf{S}\Pi_1 = \mathbf{S}^1, \quad \mathbf{S}\Pi_{2n+3} = \alpha_\omega(\mathbf{S}\Pi_{2n+1} \amalg \mathbf{S}\Sigma_{2n+1}^+).$$

For instance, $\mathbf{S}\Sigma_3 \simeq \omega+1$, $\mathbf{S}\Sigma_3^+ \simeq \mathbf{S}^1 \amalg (\omega+1)$, and $\mathbf{S}\Pi_3 \simeq \alpha_\omega(\mathbf{S}^1 \amalg 1)$. The $\mathbf{S}^1$-ranks of $\mathbf{S}\Sigma_1$, $\mathbf{S}\Pi_1$ and $\mathbf{S}\Sigma_3$ are 0, and the $\mathbf{S}^1$-ranks of $\mathbf{S}\Pi_3$ and $\mathbf{S}\Sigma_5$ are 1. In general, the $\mathbf{S}^1$-ranks of $\mathbf{S}\Pi_{2n+1}$ and $\mathbf{S}\Sigma_{2n+3}$ are n.

Theorem 3.16. *For any degree $\mathbf{d}$, there is a compact metrizable space $\mathcal{P}_{\mathbf{d},1}$ whose compact degree spectrum is $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}''\}$ and Polish degree spectrum is $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}'''\}$.*

It suffices to show the following:

Lemma 3.17. *For any $D \subseteq \omega$, there is a compact metrizable space $\mathcal{P}_{D,1}$ such that for any $Z \subseteq \omega$,*

$$D \text{ is } \Sigma^0_3 \text{ relative to } Z \iff \mathcal{P}_{D,1} \text{ has a } Z\text{-computable compact presentation.}$$
$$D \text{ is } \Sigma^0_4 \text{ relative to } Z \iff \mathcal{P}_{D,1} \text{ has a } Z\text{-computable Polish presentation.}$$

Proof. Given $D \subseteq \omega$, we define

$$\mathcal{T}_D^{n+1} = \begin{cases} \mathbf{S}^{n+1} \vee \mathbf{S}\Sigma_3 & \text{if } n \in D, \\ \mathbf{S}^{n+1} \vee \mathbf{S}\Pi_3 & \text{if } n \notin D, \end{cases}$$

Then, let $\mathcal{P}_{\mathbf{d},1}$ be the one-point compactification of the following locally compact space:

$$\coprod_{n\in\omega} \mathcal{T}_D^n \amalg \coprod_{\omega} \mathbf{S}\Pi_1.$$

As in the proof of Lemma 3.10, given a Z-computable compact presentation of $\mathcal{P}_{D,1}$, let us consider the Z-computable tree T_Z of components of $\mathcal{P}_{D,1}$. Recall that a component of $\mathcal{P}_{D,1}$ is an infinite path through T_Z. There is a unique component in $\mathcal{P}_{D,1}$ which is homeomorphic to $\mathbf{S}^{n+1}$, and let α_{n+1} be the corresponding infinite path through T_Z. At each stage s, compute a node η_s of T_Z which has a longest-surviving $(n+1)$-hole among nodes of T_Z of length s. Then, η_s is an initial segment of α_{n+1} for almost all s.

The key observation is that if $n \notin D$, then $\mathcal{T}_D^{n+1} \simeq \mathbf{S}^{n+1} \vee \mathbf{S}\Pi_3$, so $\mathbf{S}^{n+1}$ is an $\mathbf{S}^1$-rank 1 component of $\mathcal{P}_{D,1}$, that is, any neighborhood of the $(n+1)$-sphere $\mathbf{S}^{n+1}$ has a copy of $\mathbf{S}^1$ which is separated from the $(n+1)$-sphere. On the other hand, in the case $n \in D$, we have $\mathcal{T}_D^{n+1} \simeq \mathbf{S}^{n+1} \vee \mathbf{S}\Sigma_3$, so $\mathbf{S}^{2n+2}$ is an $\mathbf{S}^1$-rank 0 component of $\mathcal{P}_{D,1}$. Indeed, if a neighborhood of the $(n+1)$-sphere $\mathbf{S}^{n+1}$ is sufficiently small, then it only contains isolated points except for the sphere itself. By this observation, $n \notin D$ if and only if for any s there are $t > s$ and $\sigma \in T_Z$ such that $\sigma \succeq \eta_t$, $\sigma \not\succeq \eta_{t+1}$, and σ satisfies the $\Sigma^0_2(Z)$ property stating that σ has a 1-hole which survives forever. This is clearly a $\Pi^0_3(Z)$ property.

If $\mathcal{P}_{D,1}$ has a Z-computable Polish presentation, then by Lemma 2.1, $\mathcal{P}_{D,1}$ has a Z'-computable compact presentation. Therefore, D is Σ^0_3 relative to Z'; hence Σ^0_4 relative to Z.

For the other direction, assume that D is Σ^0_3 relative to Z. We show that $\mathcal{P}_{D,1}$ has a Z-computable compact presentation. By our assumption, there is a computable set $A \subseteq \omega^3$ such that $n \notin D$ if and only if $\exists^\infty a \forall b\, (n, a, b) \in A$. Without loss of generality, we may assume that $(n, 2a, 0) \notin A$ for any $a \in \omega$. We will construct a presentation of a space $\mathcal{P}$. First enumerate ω-many copies of $\mathbf{S}^1$ and $\omega+1$, and compactify them by adding a point z

into $\mathcal{P}$. Then we prepare for infinitely many pairwise separated regions $(R_i)_{i\in\omega}$, where R_i is 2^{-i}-close to z. Inside R_n, we start by describing $\mathbf{S}^{n+1}$. Then, prepare for a sequence of countably many separated copies $\{S^1_{n,a}\}_{a\in\omega}$ of the circle $\mathbf{S}^1$ which converge to $p \in \mathbf{S}^{n+1}$, and let $(S^1_{n,a}[s])_{s\in\omega}$ be an increasing sequence of nonempty sets of finite isolated points which 2^{-s}-approximates $S^1_{n,a}$, that is, $d_H(S^1_{n,a}[s], S^1_{n,a}) < 2^{-s}$ w.r.t. the Hausdorff distance d_H, where we do not enumerate these copies into $\mathcal{P}$ at present. For each a, we enumerate the following set into the region R_n:

$$\tilde{S}^1_{n,a} = \bigcup\{S^1_{n,a}[s] : (\forall b < s)\ (n, a, b) \in A\}.$$

In other words, if $(n, a, b) \in A$ is true for any $b < s$, then we describe a 2^{-s}-approximation of $S^1_{n,a}$ by enumerating finitely many isolated points into the region R_n; otherwise, we just keep finitely many isolated points which are already put into R_n by the previous stages. If $(n, a, b) \in A$ for all b, then $\tilde{S}^1_{n,a}$ is a dense subset of $S^1_{n,a}$, so its completion is $S^1_{n,a} \simeq \mathbf{S}^1 \simeq \mathbf{S}\Pi_1$; otherwise, $\tilde{S}^1_{n,a}$ consists of finitely many isolated points (finitely many copies of $\mathbf{S}\Sigma^+_1$), where the sentence $(\forall b < 0)\ (n, a, b) \in A$ is vacuously true, so there is at least one isolated point. Therefore, if $\exists^\infty a \forall b\, (n, a, b) \in A$ is true (i.e., $n \notin D$), then there are a sequence of circles converging to p (that is, $\alpha_\omega(\mathbf{S}\Pi_1)$), and also a sequence of isolated points converging to p (that is, $\alpha_\omega(\mathbf{S}\Sigma^+_1)$) by our assumption that $(n, 2a, 0) \notin A$ for any a. In this case, this space restricted to the region R_n is homeomorphic to $\alpha_\omega(\mathbf{S}\Pi_1) \vee \alpha_\omega(\mathbf{S}\Sigma^+_1) \simeq \alpha_\omega(\mathbf{S}\Pi_1 \amalg \mathbf{S}\Sigma^+_1) \simeq \mathbf{S}\Pi_3$. If $\exists^\infty a \forall b\, (n, a, b) \in A$ is false (i.e., $n \in D$), then there are at most finitely many circles (finitely many copies of $\mathbf{S}\Pi_1$), and a sequence of isolated points converging to p (that is, $\mathbf{S}\Sigma_3 \simeq \alpha_\omega(\mathbf{S}\Sigma^+_1)$). In this case, this space restricted to the region R_n is homeomorphic to the separated union of $\mathbf{S}^{n+1} \vee \mathbf{S}\Sigma_3$ and at most finitely many circles $\mathbf{S}\Pi_1$. Up to homeomorphism, these finitely many circles are absorbed into ω-many copies of $\mathbf{S}\Pi_1$ in some other region. Consequently, $\mathcal{P}$ is homeomorphic to $\mathcal{P}_{D,1}$. Moreover, our construction is Z-computable, and hence $\mathcal{P}_{D,1}$ has a Z-computable compact presentation.

Next, assume that D is Σ^0_4 relative to Z. We show that $\mathcal{P}_{D,1}$ has a Z-computable Polish presentation. By our assumption, there is a computable set $A \subseteq \omega^3$ such that $n \notin D$ if and only if $\exists^\infty a \exists^\infty b\, (n, a, b) \in A$. We will construct a presentation of a space $\mathcal{P}$. First enumerate ω-many copies of $\mathbf{S}^1$ and $\omega + 1$, and compactify them by adding a point z into $\mathcal{P}$. Then we prepare for infinitely many pairwise separated regions $(R_i)_{i\in\omega}$, where R_i is 2^{-i}-close to z. Inside R_n, we start by describing $\mathbf{S}^{n+1}$, and enumerating a sequence of isolated points which converges to a point $p \in \mathbf{S}^{n+1}$. Let $S^1_{n,a}[s]$ be a 2^{-s}-approximation of a copy $S^1_{n,a}$ of $\mathbf{S}^1$ as before. We enumerate the following set into the region R_n:

$$\tilde{S}^1_{n,a}[0] \cup \bigcup\{S^1_{n,a}[b] : (n, a, b) \in S\}.$$

If $(n, a, b) \in A$ for infinitely many b, then $\tilde{S}^1_{n,a}$ is a dense subset of $S^1_{n,a}$, so its completion is $S^1_{n,a} \simeq \mathbf{S}^1$; otherwise, $\tilde{S}^1_{n,a}$ consists of finitely many isolated points. Thus, by the same argument as above, $\mathcal{P}$ is homeomorphic to $\mathcal{P}_{D,1}$. Moreover, our construction is Z-computable, and hence $\mathcal{P}_{D,1}$ has an Z-computable Polish presentation. □

Theorem 3.18. *For any degree* $\mathbf{d}$*, there is a compact metrizable space* $\mathcal{P}_{\mathbf{d},2}$ *whose compact degree spectrum is* $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}^{(4)}\}$ *and Polish degree spectrum is* $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}^{(5)}\}$.

It suffices to show the following:

Lemma 3.19. *For any $D \subseteq \omega$, there is a compact metrizable space $\mathcal{P}_{D,2}$ such that for any $Z \subseteq \omega$,*

$$D \text{ is } \Sigma^0_5 \text{ relative to } Z \iff \mathcal{P}_{D,2} \text{ has a } Z\text{-computable compact presentation.}$$
$$D \text{ is } \Sigma^0_6 \text{ relative to } Z \iff \mathcal{P}_{D,2} \text{ has a } Z\text{-computable Polish presentation.}$$

Proof. Given $D \subseteq \omega$, we define

$$\mathcal{T}_D^{n+1} = \begin{cases} \mathbf{S}^{n+1} \vee \mathbf{S}\Sigma_5 & \text{if } n \in D, \\ \mathbf{S}^{n+1} \vee \mathbf{S}\Pi_5 & \text{if } n \notin D. \end{cases}$$

Then, let $\mathcal{P}_{D,2}$ be the one-point compactification of the following locally compact space:

$$\coprod_{n\in\omega} \mathcal{T}_D^{n+1} \amalg \coprod_{\omega} \mathbf{S}\Pi_3.$$

As in Lemmas 3.10 and 3.17, given a compact presentation of $\mathcal{P}_{D,2}$, we again make the Z-computable tree T_Z of components of $\mathcal{P}_{D,2}$. Recall that a component of $\mathcal{P}_{D,2}$ is an infinite path through T_Z. There is a unique component in $\mathcal{P}_{D,2}$ which is homeomorphic to $\mathbf{S}^{n+1}$, and let α_{n+1} be the corresponding infinite path through T_Z. At each stage s, compute a node η_s of T_Z which has a longest-surviving $(n+1)$-hole among nodes of T_Z of length s. Then, η_s is an initial segment of α_{n+1} for almost all s.

We then combine the idea of the proofs of Lemmas 3.10 and 3.17: The key observation is that if $n \notin D$, then $\mathcal{T}_D^{n+1} \simeq \mathbf{S}^{n+1} \vee \mathbf{S}\Pi_5$, where recall that the $\mathbf{S}^1$-rank of $\mathbf{S}\Pi_5$ is 2, so any neighborhood of $\mathbf{S}^{n+1}$ contains a component of $\mathbf{S}^1$-rank at least 1. On the other hand, in the case $n \in D$, we have $\mathcal{S}_D^{n+1} \simeq \mathbf{S}^{n+1} \vee \mathbf{S}\Sigma_5$, where recall that the $\mathbf{S}^1$-rank of $\mathbf{S}\Sigma_5$ is 1, so if a neighborhood of $\mathbf{S}^{n+1}$ is sufficiently small, then it only contains components of $\mathbf{S}^1$-rank 0 except for the sphere itself. By this observation, $n \notin D$ if and only if for any s there are $t > s$ and $\sigma \in T_Z$ such that $\sigma \succeq \eta_t$, $\sigma \not\succeq \eta_{t+1}$, and σ satisfies the $\Pi^0_3(Z)$ property stating that σ has some extension in $[T_Z]$ of $\mathbf{S}^1$-rank 1; that is, for any ℓ there are two extensions of τ of length ℓ extending satisfy the $\Sigma^0_2(Z)$ property stating that the corresponding component has a 1-hole which survives forever. In summary, the predicate $n \notin D$ is $\Pi^0_5(Z)$.

For the other direction, assume that D is Σ^0_5 relative to Z. We show that $\mathcal{P}_{D,2}$ has a Z-computable compact presentation. By our assumption, there is a computable set $A \subseteq \omega^4$ such that $n \notin D$ if and only if $\exists^\infty a \exists^\infty b \forall c\, (n, a, b, c) \in A$. Without loss of generality, we may assume that $(n, a, 0, c) \in A$ for all a, c, and $(n, a, 2b + 1, 0), (n, 2a, b + 1, 0) \notin A$ for all a, b. We will construct a presentation of a space $\mathcal{P}$. First enumerate ω-many copies of $\mathbf{S}^1$ and $\omega + 1$, and compactify them by adding a point z into $\mathcal{P}$. Then we prepare for infinitely many pairwise separated regions $(R_i)_{i\in\omega}$, where R_i is 2^{-i}-close to z.

Inside R_n, we start by describing $\mathbf{S}^{n+1}$, and enumerating a sequence of isolated points $(p_a)_{a\in\omega}$ which converges to a point $p_\infty \in \mathbf{S}^{n+1}$. That is, the following is enumerated into the region R_n:

$$\mathbf{S}^{n+1} \cup \{p_a : a \in \omega\}.$$

Then, prepare for a sequence of countably many separated copies $\{S^1_{n,a,b}\}_{b\in\omega}$ of the circle $\mathbf{S}^1$ which also converge to $p_a \in \mathbf{S}^{2n+2}$, and let $(S^1_{n,a,b}[s])_{s\in\omega}$ be an increasing sequence of nonempty sets of finite isolated points which 2^{-s}-approximates $S^1_{n,a,b}$, that is, $d_H(S^1_{n,a,b}[s], S^1_{n,a,b}) < 2^{-s}$, where we do not enumerate these copies into $\mathcal{P}$ at present. For each a, we enumerate $S^1_{n,a,0}$ and the following set into the region R_{2n}:

$$\tilde{S}^1_{n,a,b} = \bigcup\{S^1_{n,a,b}[s] : (\forall c < s)\ (n, a, b, c) \in A\}.$$

As before, if $\exists^\infty b\forall c\,(n, a, b, c) \in A$ is true, then there are a sequence of circles converging to p_a (that is, $\alpha_\omega\mathbf{S}\Pi_1$), and also a sequence of isolated points converging to p_a (that is, $\alpha_\omega\mathbf{S}\Sigma_1^+$) by our assumption that $(n, a, 2b+1, 0) \notin A$ for all b. This part is homeomorphic to $\mathbf{S}\Pi_3$ as before. If $\exists^\infty b\forall c\,(n, a, b, c) \in A$ is false, then the a-strategy enumerates at most finitely many circles (finitely many copies of $\mathbf{S}\Pi_1$), and a sequence of isolated points converging to p_a (that is, $\mathbf{S}\Sigma_3 \simeq \alpha_\omega(\mathbf{S}\Sigma_1^+)$, so it is homeomorphic to $\mathbf{S}\Sigma_3 \amalg (\mathbf{S}\Pi_1)^k$ for some k, where k is positive by our assumption that $(n, a, 0, c) \in A$ for all c.

If $\exists^\infty a\exists^\infty b\forall c\,(n, a, b, c) \in A$ is true (i.e., $n \notin D$), then there are a sequence of copies of $\mathbf{S}\Pi_3$ converging to p_∞, and also a sequence of copies of $\mathbf{S}\Sigma_3 \amalg (\mathbf{S}\Pi_1)^k$ converging to p_∞ by our assumption that $(n, 2a, b+1, 0) \notin A$ for all a, b. It is not hard to check that the union of the latter sequence and p_∞ is homeomorphic to $\alpha_\omega\mathbf{S}\Sigma_3^+$, where recall that $\mathbf{S}\Sigma_3^+ \simeq \mathbf{S}\Sigma_3 \amalg \mathbf{S}\Pi_1$. Therefore, in this case, this space restricted to the region R_{2n} is homeomorphic to the wedge sum of $\mathbf{S}^{2n+2}$ and $\alpha_\omega\mathbf{S}\Pi_3 \vee \alpha_\omega\mathbf{S}\Sigma_3^+ \simeq \alpha_\omega(\mathbf{S}\Pi_3 \amalg \mathbf{S}\Sigma_3^+) \simeq \mathbf{S}\Pi_5$.

If $\exists^\infty a\exists^\infty b\forall c\,(n, a, b, c) \in A$ is false (i.e., $n \in D$), then there are at most finitely many copies of $\mathbf{S}\Pi_3$ and a sequence of copies of $\mathbf{S}\Sigma_3 \amalg (\mathbf{S}\Pi_1)^k$ converging to p_∞. The latter part is homeomorphic to $\alpha_\omega\mathbf{S}\Sigma_3^+ \simeq \mathbf{S}\Sigma_5$ as above. Therefore, in this case, this space restricted to the region R_{2n} is homeomorphic to the separated union of $\mathbf{S}^{2n+2} \vee \mathbf{S}\Sigma_5$ and at most finitely many copies of $\mathbf{S}\Pi_3$. Up to homeomorphism, these finitely many copies of $\mathbf{S}\Pi_3$ are absorbed into ω-many copies of $\mathbf{S}\Pi_3$ in some other region.

Consequently, $\mathcal{P}$ is homeomorphic to $\mathcal{P}_{D,2}$. Moreover, our construction is $\mathbf{x}$-computable, and hence $\mathcal{P}_{D,2}$ has a Z-computable compact presentation. For a Polish presentation, the similar argument as above works. □

Theorem 3.20. *For any degree $\mathbf{d}$ and $n > 0$, there is a compact metrizable space $\mathcal{P}_{\mathbf{d},n}$ whose compact degree spectrum is $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}^{(2n)}\}$ and Polish degree spectrum is $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}^{(2n+1)}\}$.*

It suffices to show the following:

Lemma 3.21. *For any $D \subseteq \omega$ and $n > 0$, there is a compact metrizable space $\mathcal{P}_{D,n}$ such that for any $Z \subseteq \omega$,*

$$D \text{ is } \Sigma^0_{2n+1} \text{ relative to } Z \iff \mathcal{P}_{D,n} \text{ has a } Z\text{-computable compact presentation.}$$
$$D \text{ is } \Sigma^0_{2n+2} \text{ relative to } Z \iff \mathcal{P}_{D,n} \text{ has a } Z\text{-computable Polish presentation.}$$

Proof. Given $D \subseteq \omega$, we define

$$\mathcal{T}_D^{d+1} = \begin{cases} \mathbf{S}^{d+1} \vee \mathbf{S}\Sigma_{2n+1} & \text{if } d \in D, \\ \mathbf{S}^{d+1} \vee \mathbf{S}\Pi_{2n+1} & \text{if } d \notin D, \end{cases}$$

Then, let $\mathcal{P}_{D,n}$ be the one-point compactification of the following locally compact space:

$$\coprod_{d\in\omega}\mathcal{T}_D^{d+1}\amalg\coprod_{\omega}\mathbf{S}\Pi_{2n-1}.$$

Then, proceed the similar argument as above. $\square$

As a corollary, for any natural number n, the class of high$_{n+1}$-degrees is a compact degree spectrum of a compact metrizable space, and the class of high$_{2n+3}$-degrees is a Polish degree spectrum of a compact metrizable space.

4 Cone-avoidance

In this section, we solve Question 3 by showing the following:

Theorem 4.1. *Let $A \subseteq \omega$ be a non-c.e. set. Every compact Polish space has a Polish presentation that does not enumerate A.*

In particular,

Corollary 4.2. *The degree spectrum of a compact Polish space cannot be the upper cone $\{\mathbf{x} : \mathbf{d} \leq \mathbf{x}\}$ for any non-computable degree $\mathbf{d}$.*

Actually the proof also shows that if for each i we choose a non-c.e. set A_i, then every compact Polish space has a presentation that does not enumerate any A_i. It implies that the degree spectrum of a compact Polish space cannot be a countable union of non-trivial upper cones $\bigcup_{i\in\omega}\{\mathbf{x} : \mathbf{d}_i \leq \mathbf{x}\}$.

In order to prove the result, we need ideas from computability theory and ideas from topology.

Overtness argument. Overtness captures a familiar argument in computably theory, which we describe now.

We will apply the technique to the space $X = \mathcal{V}(Q)$, however it is easier to state the result for an abstract space X.

Let X be a countably-based space with a fixed indexed basis $(B_i)_{i\in\omega}$ that is closed under finite intersections. We say that A is a reducible to x if A is enumeration reducible to $N_x = \{i \in \omega : x \in B_i\}$. We write $M(x) = A$ if M enumerates A from any enumeration of N_x. We say that a set $S \subseteq X$ is computably overt if the set $\{i \in \omega : S \cap B_i \neq \emptyset\}$ is c.e.

Given a Turing machine M and a set A, we say that M fails to enumerate A from x if M outputs some $n \notin A$ on some enumeration of N_x. We denote by $F_{M,A}$ the set of x's on which M fails to enumerate A. That set is open, because when M outputs some $n \notin A$, it has only read a finite part of N_x, which can be extended to an enumeration of N_y for any y in some neighborhood of x, so each such y also belongs to $F_{M,A}$.

Lemma 4.3. *Let X be a countably-based space. Let $A \subseteq \omega$ be a non-c.e. set. If $x \in X$ enumerates A using machine M, then $F_{M,A}$ intersects every computably overt set containing x.*

Proof. Assume that $M(x) = A$ and let V be a computably overt set containing x. If $F_{M,A}$ does not intersect V, then we describe an effective procedure that enumerates A, contradicting the assumption that A is not computable. The procedure is as follows: enumerate all the prefixes of names of elements of V (which is possible because V is computably overt), simulate M on them, and collect all the outputs. As $F_{M,A}$ does not intersect V, all the outputs are correct, i.e. belong to A, and every element of A appears because M enumerates A on each name of $x \in V$. As a result, this procedure enumerates A, which is then c.e. The contradiction implies that $F_{M,A}$ intersects V. $\square$

Perturbations. We now come to the topological ingredient of the proof, based on the notion of ϵ-perturbation. The Hilbert cube Q is endowed with the complete metric

$$d(x, y) = \sum_i 2^{-i}|x_i - y_i|.$$

The proof is a Baire category argument: one can perturb any compact subset of Q so that its perturbed copy does not enumerate A.

Definition 4.4. An **ϵ-perturbation** is a one-to-one continuous function $f : Q \to Q$ such that $d(f(x), x) < \epsilon$ for all $x \in Q$.

Lemma 4.5. *Let $S = \{s_0, \dots, s_n\}$ and $T = \{t_0, \dots, t_n\}$ be subsets of Q such that $d(s_i, t_i) < \epsilon$ for $i = 0, \dots, n$. There exists an ϵ-perturbation f such that $f(s_i) = t_i$ for $i = 0, \dots, n$.*

Proof. It is a corollary of the homeomorphism extension theorem (Theorem 5.3.7 in [26]), stating that if S, T are Z-sets and $f : S \to T$ is a homemorphism satisfying $d(f(x), x) < \epsilon$ for all $x \in S$, then f can be extended to a homeomorphism $\overline{f} : Q \to Q$ satisfying the same condition for all $x \in Q$. We only need to know that finite sets are Z-sets: in the same reference, singletons are Z-sets by Remark 5.1.4, and finite unions of Z-sets are again Z-sets by Lemma 5.1.2. $\square$

We remind the reader that $\mathcal{V}(Q)$ is a topological space endowed with the lower Vietoris topology. In the next statement, the notions of computable overtness and closure are meant in that topology.

Lemma 4.6. *Let $D \in \mathcal{V}(Q)$ and $\epsilon > 0$. There exists a computably overt set $\mathcal{A} \subseteq \mathcal{V}(Q)$ containing D and contained in the closure of the set of ϵ-deformations of D.*

Proof. Let $\mathcal{F} \subseteq \mathcal{V}(Q)$ be the family of finite sets of rational points, which can be indexed in an obvious way. We are going to define $\mathcal{A}$ in such a way that:

(1) $\mathcal{F} \cap \mathcal{A}$ is dense in $\mathcal{A}$,

(2) $\mathcal{F} \cap \mathcal{A}$ is computably enumerable,

(3) Every element of $\mathcal{F} \cap \mathcal{A}$ is contained in some ϵ-deformation of D.

The first two conditions imply that $\mathcal{A}$ is computably overt, because it is the closure of a computable dense sequence.

The first and third conditions imply that $\mathcal{A}$ is contained in the closure of the set of ϵ-deformations of D: $\mathcal{A}$ is contained in the closure of $\mathcal{F} \cap \mathcal{A}$, and each element of $\mathcal{F} \cap \mathcal{A}$ is a subset of an ϵ-deformation C of D, so belongs to the closure of $\{C\}$ (the lower Vietoris closed open sets are upwards closed, equivalently the lower Vietoris closed sets are downwards closed).

We now define $\mathcal{A}$ satisfying these conditions. If D was perfect then we could just take some small ball around D in the Hausdorff metric. However we need a bit more work in the general case.

We first show that there exist open sets $U_0, \dots, U_n \subseteq Q$ that cover D, such that for every $x \in U_0$, $D \cap B(x, \epsilon)$ is infinite and $D \cap U_i$ is a singleton for each $i \geq 1$. Let D_N be the set of non-isolated points of D and let $U_0 = D_N^\epsilon$. The set $D \setminus U_0$ is finite, because it is compact and all its points are isolated. Therefore, there exist basic balls $U_1, \dots, U_n$ isolating the points of $D \setminus U_0$. We can make sure that the open sets U_i are pairwise disjoint. By compactness of D, we can assume that U_0 is a finite union of basic balls. We can now define our computably overt subset of $\mathcal{V}(Q)$: let

$$\mathcal{A} = \left\{ E \in \mathcal{V}(Q) : E \subseteq \bigcup_{0 \leq i \leq n} U_i \text{ and } |E \cap U_i| = 1 \text{ for all } i \geq 1 \right\}.$$

Conditions 1. and 2. are easily checked.

We now prove condition 3.

Claim 1. For every finite set $T \in \mathcal{A}$, there exists and ϵ-deformation of D containing T.

Let $T = \{t_0, \dots, t_n\}$ belong to $\mathcal{A}$. We build a finite set $S = \{s_0, \dots, s_n\} \subseteq D$ with $d(s_i, t_i) < \epsilon$. For each i, if $t_i \in U_0$ then the intersection of $B(t_i, \epsilon)$ with D is infinite, so we can choose a point s_i in the intersection, so that $s_i \neq s_j$ for $i \neq j$. If $t_i \in U_k$ with $k \geq 1$, then we define s_i as the unique point of D in U_k. The points s_i are pairwise distinct, because if $i \neq j$ then s_i and s_j cannot both belong to a common U_k, $k \geq 1$, as $D \in \mathcal{A}$.

One has $d(s_i, t_i) < \epsilon$ for each i, so we can apply Lemma 4.5 to obtain an ϵ-perturbation f mapping each s_i to t_i. One has $T \subseteq f(D)$ so the claim is proved, and the Lemma as well. □

We now have all the ingredients needed to prove the result.

Proof of Theorem 4.1. Let X be a compact Polish space. We prove that some copy of X in $\mathcal{V}(Q)$ does not enumerate A. It implies the result, because any name of a copy of X in $\mathcal{V}(Q)$ computes a presentation of X.

The space $\mathcal{F} = \{\phi : Q \to Q \text{ continuous one-to-one}\}$, with the topology induced by the metric $d(\phi, \psi) = \sup_x d(\phi(x), \psi(x))$ is Polish.

For any compact set $C \subseteq Q$ and any non-c.e. set $A \subseteq \omega$, we prove that the set $\{\phi \in \mathcal{F} : A \leq_e \phi(C)\}$ is meager in $\mathcal{F}$, which implies the existence of a copy D of C such that $A \not\leq_e D$. It is done by showing that for each Turing machine M, the set $\{\phi \in \mathcal{F} : M(\phi(C)) = A\}$ is nowhere dense in $\mathcal{F}$.

Let $\phi \in \mathcal{F}$ be such that $M(\phi(C)) = A$. For $\epsilon > 0$, we prove that there exists $\psi \in \mathcal{F}$ such that $d(\phi, \psi) < \epsilon$ and such that $\psi(C) \in F_{M,A}$. It implies the result, because for every ψ' sufficiently close to ψ, one also has $\psi' \in F_{M,A}$ as $F_{M,A}$ is open.

Lemma 4.6 provides a computably overt set $\mathcal{A}$ containing $\phi(C)$, in which the set of ϵ-deformations of $\phi(C)$ is dense. By Lemma 4.3, $F_{M,A}$ intersects $\mathcal{A}$. As $F_{M,A}$ is open, there exists an ϵ-deformation of $\phi(C)$ in $F_{M,A}$. Let f be the corresponding ϵ-perturbation, and $\psi = f \circ \phi$. One has $d(\psi, \phi) < \epsilon$ and $\psi(C) \in F_{M,A}$. □

5 Comparing compact and Polish presentations

Let X be a compact Polish space. In the proofs we frequently use the fact that the jump of any Polish presentation of X computes a compact presentation of X, which is stated precisely in Lemma 2.1. Here we investigate whether it can compute more. Of course, it always computes $\mathbf{0}'$, and we show that if X is perfect, then it does not compute more in general: every compact presentation of X, paired with $\mathbf{0}'$, computes the jump of a Polish presentation of X. However, it is no more true for non-perfect spaces and we give a counter-example.

Theorem 5.1. *Let X be a compact perfect Polish space. The jumps of the Polish presentations of X are the compact presentations of X paired with $\mathbf{0}'$.*

In other words, one has $\{\mathbf{d}' : X \text{ has a } \mathbf{d}\text{-computable Polish presentation}\} = \{(\mathbf{d}, \mathbf{0}') : X \text{ has a } \mathbf{d}\text{-computable compact presentation}\}$.

Reformulation. Again, we use some overtness argument to reformulate the problem.

We can reformulate the jumps of the Polish presentations of X, i.e. the degrees $\mathbf{d}'$ such that X has a $\mathbf{d}$-computable Polish presentation.

As we have already seen the degrees of Polish presentations of X coincide with the degrees of copies of X as elements of $\mathcal{V}(Q)$. In the same way, the jumps of these degrees are exactly the jumps of the copies of X in $\mathcal{V}(Q)$.

Again, we abstract away from $\mathcal{V}(Q)$ to make the results easier to read. Let S be an effective countably-based space with a fixed index basis $(B_i)_{i\in\omega}$ that is closed under finite intersections. Let $(U_i^S)_{i\in\omega}$ be an effective enumeration of the effective open subsets of S: $U_i^S = \bigcup_{j\in W_i} B_j$.

Definition 5.2. The jump of $x \in S$ is the set $J(x) = \{i \in \omega : x \in U_i^S\}$.

Lemma 5.3. *In an effective countably-based space S, computing $J(x)$ is equivalent to computing $\mathbf{d}'$ for some $\mathbf{d}$ that computes x.*

Proof. Let $\delta_S :\subseteq \omega^\omega \to S$ be the standard representation of S, mapping p to x if $\{i : \exists n, p(n) = i+1\} = \{i : x \in \mathcal{B}_i\}$, in which case we say that p is a name of x. The function δ_S is computable and effectively open: the image of an effective open is an effective open set, uniformly. Observe that $\mathbf{d}$ computes x if and only if $\mathbf{d}$ computes some name of x.

Because δ_S is computable, the preimages of effective open sets are effectively open, so if p is a name of x then p' computes $J(x)$.

Conversely, given $J(x)$, we show how to compute p' for some name p of x. Let $(U_n)_{n\in\omega}$ be the canonical enumeration of the effective open subsets of ω^ω. Let V_n be x-effective open sets such that $\delta_S^{-1}(x) = \bigcap_n V_n$.

At stage s, we have produced a finite prefix p_s of p. Given $J(x)$, we can decide whether $[p_s] \cap \delta_X^{-1}(x)$ intersects U_s, because it is equivalent to $x \in \delta_S([p_s] \cap U_s)$ which is an effective open set for which we have an index. If it does, then we extend p_s to p_{s+1} so that $[p_{s+1}] \subseteq U_s \cap V_s$. If it does not, then we simply make sure that $[p_{s+1}] \subseteq V_s$.

In the limit, we obtain some $p \in \bigcap_s V_s$ so p is a name of x. For each s, we have decided along the construction whether $p \in U_s$, so we have computed p'. $\square$

Proof of Theorem 5.1. We are given $C \in \mathcal{K}(Q)$, together with $\mathbf{0}'$. We progressively compute a copy D of C, together with its jump as a point of $\mathcal{V}(Q)$. Let $(\mathcal{U}_n)_{n\in\omega}$ be an effective enumeration of the effective open subsets of $\mathcal{V}(Q)$. For each n, we need to decide as long as we build D, whether $D \in \mathcal{U}_n$.

We start from some $\epsilon > 0$ and some basic $\epsilon/2$-ball $\mathcal{B}_0$ containing C. It is a computably overt set in $\mathcal{V}(Q)$, so we can decide using $\mathbf{0}'$ whether it intersects $\mathcal{U}_0$. There are two cases. In the first case, $\mathcal{B}_0$ intersects $\mathcal{U}_0$. As in Lemma 4.6 there exists an ϵ-perturbation f_0 mapping C to $C_0 \in \mathcal{U}_0$ (as C is perfect, the computably overt set given by Lemma 4.6 can be replaced by the $\epsilon/2$-ball $\mathcal{B}_0$).

Claim 2. We can compute such an f_0.

The space $\mathcal{P}_\epsilon$ of ϵ-perturbations is a computable Polish space, the function $\Phi : \mathcal{P}_\epsilon \to \mathcal{V}(Q)$ mapping f to $f(C)$ is C-computable, so $\Phi^{-1}(\mathcal{U}_0)$ is a C-effective open set, in which we can computably find some f_0, so the claim is proved.

We pick a ball around C_0, whose closure is contained in $\mathcal{U}_0$ and in which we are going to stay forever, so that in the limit, the copy D of C belongs to $\mathcal{U}_0$. We declare that $D \in \mathcal{U}_0$.

The second case is if $\mathcal{B}_0$ does not intersect $\mathcal{U}_0$. In that case, we do nothing and proceed. In the sequel, we stay forever in $\mathcal{B}_0$ (and even in some closed ball contained in $\mathcal{B}_0$) so that in the limit, $D \in \mathcal{B}_0$ hence D does not belong to $\mathcal{U}_0$. We declare that $D \notin \mathcal{U}_0$.

In both cases, we have decided whether the set D belongs to $\mathcal{U}_0$. We now iterate this process with $\mathcal{U}_1, \mathcal{U}_2$, etc., taking ϵ smaller and smaller so that the composition of the ϵ-perturbations converges to a homeomorphism (the Inductive Convergence Criterion [26] tells us that we can always choose the next ϵ sufficiently small to ensure that the limit is a homeomophism, and moreover ϵ can be chosen in a computable way), and taking the closure of $\mathcal{B}_{n+1}$ contained in $\mathcal{B}_n$. In the limit, we have built a copy D of C and computed its jump. $\square$

Observe that the argument is uniform, assuming that the space is perfect. We now observe that there cannot exist a uniform argument including non-perfect Polish spaces. Indeed, whether X is not perfect is Σ^0_2 for a Polish presentation, so it is Σ^0_1 in its jump. If there was a uniform argument then being non-perfect would be $\Sigma^0_1(\mathbf{0}')$ for compact presentations, in particular it would be open. However, the set of non-perfect compact sets is not open in the Hausdorff metric (witnessed for instance by a sequence of segments shrinking to a singleton).

We actually show that Theorem 5.1 simply does not extend to non-perfect Polish spaces. Observe that a corollary of Theorem 5.1 is that if X is a perfect Polish space with a $\mathbf{0}'$-computable compact presentation, then it has a low Polish presentation. We show that it fails for some non-perfect Polish space.

Proposition 5.4. *There exists a (non-perfect) compact Polish space with a $\mathbf{0}'$-computable compact presentation, but no low Polish presentation. Therefore, it has a compact presentation $\mathbf{d}$ such that $(\mathbf{d}, \mathbf{0}')$ does not compute the jump of any Polish presentation of it.*

Proof. Let $\mathbf{d} = \mathbf{0}^{(6)}$. The space $X = \mathcal{Z}_{\mathbf{d},3}$ from Theorem 3.11 has a $\mathbf{0}'$-computable compact presentation and its Cantor-Bendixon derivative $X' \cong \mathcal{Z}_{\mathbf{d},2}$ has no $\mathbf{0}''$-computable compact presentation.

Lemma 2.5 implies in particular that if X had a low Polish presentation then X' would have a $\mathbf{0}''$-computable compact presentation, which is not the case. $\square$

6 Open Questions

One of the main questions of this topic (which is probably very hard to answer completely, as also for algebraic structures) is to characterise the spectra of Polish spaces, say by proving its coincidence with the spectra of a natural class of algebraic structures. However, we still do not know the answer to the following question.

Question 4. *Is any Polish (compact) degree spectrum of a Polish space a degree spectrum of an algebraic structure?*

Recall also that one of the key ideas of our constructions is using *dimension* (more explicitly, *high-dimensional holes*, i.e., a cycle which is not a boundary) to code a given Turing degree. As a result, all of our examples in the above results are *infinite dimensional*. We do not know if there are finite dimensional examples satisfying our main results. We also note that all of our examples are disconnected, and it is not known if there are connected examples.

Question 5. *Does there exist a finite dimensional (connected) low$_4$-presented Polish space which is not homeomorphic to a computably presented one?*

There are many other open questions. For instance, the following is also open.

Question 6. *Does there exist a low$_2$-presented Polish space which is not homeomorphic to a computably presented one?*

The full solution to Question 3 is also yet to be known.

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